4 Infinite Series - I.

Given a sequence $\{a_n\}$ in $\mathbb{C}$ we define the sequence of partial sums $\{S_n\}$ by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n .$$

Def^n: We say that the infinite series $\sum a_n$ converges if $\{S_n\}$ converges, and we then denote the sum of the infinite series by

$$S = \sum_{k=1}^{\infty} a_k .$$

Else we say that the series diverges.

Th^m 4.1 (Cauchy criteria) A series $\sum a_n$ converges if and only if $\forall\ \varepsilon > 0$, $\exists\ N$ s.t

$$\begin{matrix} n, m > N \\ n \leq m \end{matrix} \implies \left| \sum_{k=n}^{m} a_k \right| < \varepsilon$$

Pf: $\mathbb{C}$ is complete. So $\sum a_n \hookrightarrow \overset{def^n}{\Longleftrightarrow} \{S_n\} \hookrightarrow$

$\Longleftrightarrow \{S_n\}$ is Cauchy $\Longleftrightarrow \forall\ \varepsilon > 0$, $\exists\ N$ s.t

$n, m > N \Rightarrow |S_n - S_m| < \varepsilon \Longleftrightarrow \left| \sum_{k=n}^{m} a_k \right| < \varepsilon$

Cor 4.2: (Divergence Test) If $\sum a_n$ converges then

$$\lim_{n\to\infty} a_n = 0$$

$\Leftrightarrow$ If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

Pf: $\forall\ \varepsilon > 0$, $\exists\ N$ s.t. $\forall\ n > N$.

$$\left| \sum_{k=n}^{n} a_k \right| < \varepsilon \Rightarrow |a_n| < \varepsilon.$$

$$\Rightarrow \quad a_n \to 0$$

- Examples 1) Telescoping series: Consider

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

Note that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$.

So $$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left[\frac{1}{k} - \frac{1}{k+1} \right]$$

$$= \left(1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1} \xrightarrow{n\to\infty} 1.$$

So the series converges and infact

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

More generally:

<u>Th^m</u> 4.3. If $a_n \in \mathbb{C}$, then the series $\sum (a_n - a_{n+1})$ converges if and only if $a_n \to 0$, and then

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1.$$

2) <u>Geometric Series</u>: Let $z \in \mathbb{C}$ and consider

$$\sum_{n=0}^{\infty} z^n.$$

It is easy to see

$$S_n = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}.$$

Clearly if $|z| < 1$ then $S_n \to 1/1-z$

$|z| > 1$ then $|z^n| \to \infty$ so divergence!

If $|z| = 1$ it is unclear. For now, we state

<u>Th^m 4.4</u>: The series $\sum z^n$ converges if $|z| < 1$ & diverges if $|z| > 1$. Moreover, if $|z| < 1$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$

· <u>Comparison Test</u>

<u>Th^m 4.5</u> If $a_n, b_n \geq 0$; and $c > 0$ s.t

$$a_n \leq c \cdot b_n \quad \forall\, n > N_0.$$

Then $\sum b_n$ converges $\Longrightarrow$ $\sum a_n$ converges

We need the foll. observation

<u>Lemma</u> 4.6 If $a_n \geq 0$, then $\sum a_n$ converges if and only if $\{S_n\}$, the seq. of partial sums is bounded.

<u>Pf</u>: Since $a_n \geq 0$, S_n is monotonic. So $\{S_n\} \to$ if and only if it is bounded.

<u>Pf of Th^m 4.5</u> If $S_n = \sum_{k=1}^{n} a_n$, $t_n = \sum_{k=1}^{n} b_n$

Then $S_n \leq c t_n$.

$\sum b_n$ converges $\Longrightarrow$ t_n is bounded

$\Longrightarrow$ S_n is bounded

$\overset{\text{Lemma}}{\Longrightarrow}$ $\sum a_n$ converges

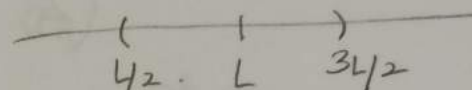
<u>Th^m</u> 4.6 (Limit Comparison test). Assume $a_n, b_n \geq 0$. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \neq 0$$

Then $\sum a_n$ converges if and only if

Pf: $\exists\ N$ s.t $\forall\ n \geq N$

$$\frac{1}{2}L < \left|\frac{a_n}{b_n}\right| < \frac{3}{2}L$$



since $L \neq 0$. Since $a_n ; b_n \geq 0$

$$\frac{L}{2} \cdot b_n < a_n < \frac{3L}{2} \cdot b_n \quad \text{or} \quad \begin{cases} a_n < \frac{3L}{2} b_n \\ b_n < \frac{2}{L} a_n \end{cases}$$

So by Th^m 4.5, $\sum a_n$ converges if $\sum b_n$ converg
$\sum b_n$ converges if $\sum a_n$ converg

Rk: If $L = 0$, then the argument shows

$$\sum b_n \hookrightarrow \ \Rightarrow\ \sum a_n \hookrightarrow$$

If $L = \infty$, then $\sum a_n \hookrightarrow\ \Rightarrow\ \sum b_n$ converges.

To apply this we need a "toolkit" of converging serie

- p-series: Consider $1 + \frac{1}{2} + \frac{1}{3} + \cdots$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots \frac{1}{15} + \frac{1}{16}\right) + \cdots$$

$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots$$

$$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent!

On the other hand consider

$$1+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right)+\left(\frac{1}{8^2}+\cdots\right.$$

$$\leq 1+2\cdot\frac{1}{2^2}+4\cdot\frac{1}{4^2}+8\cdot\frac{1}{8^2}+\cdots$$

$$\leq 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

$$=\sum_{j=0}^{\infty}\frac{1}{2^j}=\frac{1}{1-1/2}=2.$$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

More generally

<u>Th^m 4.7</u> $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$.

Generally proved using integral test. We provide alternate method.

<u>Th^m 4.8</u> Sps $a_1 \geq a_2 \geq \cdots \geq 0$ i.e. $a_n \downarrow$. Then $\sum a_n$ converges if and only if

$$\sum_{k=0}^{\infty}2^k\, a_{2^k} = a_1+2a_2+4a_4+8a_8+\cdots$$

converges.

Pf: Suffices to show partial sums are bounded

Define
$$S_n = a_1 + a_2 + \cdots + a_n$$
$$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}.$$

Claim: If $n = 2^k$

$$\frac{t_k}{2} \leq S_n \leq t_k$$

Pf:
$$S_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + a_{2^k}$$
$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1} a_{2^{k-1}}$$
$$+ a_{2^k}.$$
$$\leq a_1 + 2a_2 + \cdots + 2^{k-1} a_{2^{k-1}} + 2^k a_{2^k}$$
$$= t_k.$$

On the other hand
$$S_n = a_1 + a_2 + (a_3 + a_4) + a_5 + a_6 + a_7 + a_8$$
$$+ (\cdot + a_{2^k})$$
$$\geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots 2^{k-1} a_{2^k}$$
$$= \frac{1}{2}(2a_1 + 2a_2 + 4a_4 + \cdots 2^k a_{2^k}).$$
$$\geq \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots 2^k a_{2^k})$$
(since $a_1 \geq 0$)
$$= \frac{t_k}{2}.$$

By claim $\{S_n\}$ is bdd $\Longleftrightarrow$ $\{t_k\}$ is bounded.

Done!

<u>Pf of Th^m 4.7</u>: Let $a_n = 1/n^p$. Then

$$\sum 2^k a_{2^k} = \sum 2^k \cdot \frac{1}{2^{kp}} = \sum \frac{1}{2^{k(p-1)}}$$

which is a geometric series which converges if $\frac{1}{2^{p-1}} < 1$ or $p > 1$.

and diverges if $\frac{1}{2^{p-1}} > 1$ or $p < 1$

At $p = 1$ it is $\sum 1$ which diverges.

<u>Examples</u>: 1) $\sum_{n=1}^{\infty} \frac{1}{2n^2+n}$, $a_n = (2n^2+n)^{-1}$
$b_n = (n^2)^{-1}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2} \neq 0$

So $\sum \frac{1}{2n^2+n}$ converges since $\sum \frac{1}{n^2}$ converges.

2) $\sum_{n=1}^{\infty} \frac{1}{n\log n}$, $a_n = \frac{1}{n\log n}$

$$2^k a_{2^k} = \frac{2^k}{2^k \cdot \log 2^k} = \frac{1}{k\log 2}$$

So $\sum 2^k a_{2^k} = \frac{1}{\log 2} \sum \frac{1}{k}$ diverges

$\Rightarrow \sum \frac{1}{n\log n}$ diverges

- Absolute & Conditional Conv

Th^m 4.9 $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges.

Pf: $\sum |a_n|$ conv $\Rightarrow$ $\forall\ \varepsilon > 0$, $\exists$ N s.t

$$n, m > N \Rightarrow \sum_{k=n}^{m} |a_k| < \varepsilon.$$

But then

$$\left| \sum_{k=n}^{m} a_k \right| \overset{\Delta\text{-ineq}}{\leq} \sum_{k=n}^{m} |a_k| < \varepsilon. \quad \forall\ n, m > N.$$

$\underset{\text{criteria}}{\overset{\text{Cauchy}}{\Longrightarrow}}$ $\sum a_n$ conv.

Def^n: $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges.

If $\sum a_n$ converges but $\sum |a_n|$ does not, we say $\sum a_n$ is conditionally conv.

Example: Consider $\sum \cos(n)/n\sqrt{n}$.

$a_n = \dfrac{\cos(n)}{n\sqrt{n}}$, $|a_n| < \dfrac{1}{n\sqrt{n}}$, $\sum \frac{1}{n\sqrt{n}}$ conv

$\overset{\text{comparison}}{\Longrightarrow}$ $\sum |a_n|$ conv.

So $\sum \cos(n)/n\sqrt{n}$ converges & is in fact absolutely conv.

· <u>Ratio test</u>

<u>Th</u>^{m} 4.10 The series $\sum a_n$.

(a) converges absolutely if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(b) diverges if $\exists\ n_0$ s.t

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \quad \forall\ n \geq n_0.$$

<u>Pf</u>: (a). $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \exists\ \beta < 1$ & N s.t

$$\forall\ n \geq N, \qquad \left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

$$|a_{N+1}| < \beta |a_N|.$$
$$|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|.$$

In general $\quad |a_{N+k}| < \beta^k |a_N|$.

$\sum \beta^k$ converges $\Rightarrow \sum_{k=0}^{\infty} |a_{N+k}|$ converges.

$\Rightarrow |a_1| + |a_2| + \cdots |a_N| + \sum_{k=0}^{\infty} |a_{N+k}|$ converges.

$\Rightarrow \sum a_n$ converges absolutely.

(b) $|a_{n+1}| \geq |a_n| \geq |a_{n-1}| \geq \cdots |a_{n_0}|$.

i.e $\forall n \geq n_0$, $|a_n| \geq |a_{n_0}|$.

So $\lim_{n\to\infty} a_n \neq 0$.

$\Rightarrow \sum a_n$ diverges.

<u>Example</u>: Consider $\sum_{n=0}^{\infty} \frac{1}{n!}$ $(0! = 1)$.

$a_n = 1/n!$ Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \xrightarrow{n\to\infty} 0 < 1$.

$\Rightarrow \sum \frac{1}{n!}$ converges.

<u>Def^n</u>: We define the number 'e' to be

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

<u>Th^m</u> 4.11 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

<u>Pf</u>: Let $S_n = \sum_{k=1}^{n} \frac{1}{k!}$, $t_n = \left(1+\frac{1}{n}\right)^n$.

Binomial $\Rightarrow$ $t_n = 1 + 1 + \binom{n}{2}\cdot\frac{1}{n^2} + \binom{n}{3}\cdot\frac{1}{n^3} + \cdots \binom{n}{n}\cdot\frac{1}{n^n}$.

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} = S_n .$$

$$\Rightarrow \quad \limsup_{n\to\infty} t_n \leq S_n .$$

$$\Rightarrow \quad \limsup_{n\to\infty} t_n \leq e .$$

Next if $n \geq m$.

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)$$

Let $n \to \infty$, keeping m fixed.

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!}$$

$$\Rightarrow \quad \liminf_{n\to\infty} t_n \geq S_m \quad \forall m .$$

So $\quad \liminf_{n\to\infty} t_n \geq e .$

But $\quad \liminf \leq \limsup$

$$\Rightarrow \quad \liminf t_n = \limsup t_n = e .$$

$$\Rightarrow \quad \lim_{n\to\infty} t_n = e .$$

· <u>Root test</u>:

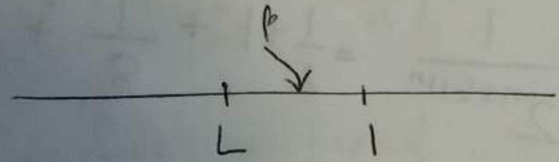
Th^m 4.12 Given $\sum a_n$, let $L = \limsup_{n\to\infty} |a_n|^{1/n}$.

(a) $L < 1 \Rightarrow \sum a_n$ conv. absolutely.

(b) $L > 1 \Rightarrow \sum a_n$ diverges.

(c) $L = 1 \Rightarrow$ inconclusive.

Pf: (a)



Let $\beta \in (L, 1)$.

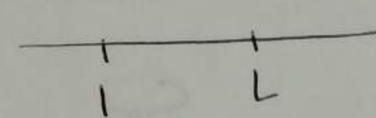
Then $\exists$ N s.t $\forall$ $n > N$

$$|a_n|^{1/n} < \beta$$

$$\Rightarrow \quad |a_n| < \beta^n.$$

$$\sum \beta^n \text{ conv} \xRightarrow[\text{test}]{\text{Comparison}} \sum |a_n| \text{ conv.}$$

(b) $L > 1 \Rightarrow \exists$ $n_k \to \infty$ s.t



$$|a_{n_k}|^{1/n_k} > 1$$

$$\Rightarrow \quad |a_{n_k}| > 1$$

$$\Rightarrow \quad \lim_{n\to\infty} |a_n| \neq 0.$$

$$\Rightarrow \quad \sum a_n \text{ diverges}$$

(c) $a_n = 1/n^2$, $b_n = 1/n$.

$$\lim_{n\to\infty} |a_n|^{1/n} = 1 = \lim_{n\to\infty} |b_n|^{1/n}$$

$\sum a_n$ conv. but $\sum b_n$ diverges.

<u>Example</u> (Rearranged geometric series)

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+(-1)^n}} = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} \cdots$$

$$a_n = \frac{1}{2^{n+(-1)^n}} = \begin{cases} \dfrac{1}{2^{n-1}} & n \text{ is odd} \\ \dfrac{1}{2^{n+1}} & n \text{ is even.} \end{cases}$$

$$|a_n|^{1/n} = \frac{1}{2^{1+(-1)^n/n}} \xrightarrow{n\to\infty} \frac{1}{2} < 1.$$

So $\sum a_n$ converges absolutely.

If we instead applied ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 1/8 & n \text{ is odd} \\ 2 & n \text{ is even} \end{cases}$$

So $$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2 > 1$$

But there are infinitely many n for which $$\left|\frac{a_{n+1}}{a_n}\right| < 1.$$

So ~~root~~ ratio test in conclusive.

<u>Rk</u>: In general root test is stronger. One can show

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \liminf_{n\to\infty} |a_n|^{1/n} < \limsup_{n\to\infty} |a_n|^{1/n}$$
$$< \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \limsup_{n\to\infty} |a_n|^{1/n} < 1$.

So if ratio test works then root test works.

But as the example suggests there are instances of root test working but ratio test failing.