- Exponentials, log and trig functions

1) Exp(x): Consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ratio test $\Longrightarrow$ Series converges for all $x \in \mathbb{R}$

Defⁿ: We define the exponential function

$$e^x : \mathbb{R} \to \mathbb{R}$$

by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

FACT (to be proved at the end of the course).

1) e^x is a diff function on $\mathbb{R}$.

2) $\frac{d}{dx} e^x = e^x \quad \forall x \in \mathbb{R}$.

Rk: For 2) Note that if we diff each term in the infinite serie

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

So 2) says that term-by-term diff. is valid.

$\underline{\text{Th}^m}$ 1) $e^0 = 1$, $e^{-x} = 1/e^x$

2) Ae^x is the <u>unique</u> function $f: \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} f' = f \\ f(0) = A \end{cases}$$

3) $e^{x+y} = e^x \cdot e^y$

4) $e^x \neq 0 \;\; \forall\, x \in \mathbb{R}$, and hence $e^x > 0 \;\; \forall\, x \in \mathbb{R}$

5) e^x is strictly monotonically increasing, $\lim_{x \to \infty} e^x = \infty$

$\underline{\text{Pf}}$: 1) $e^0 = 1$ trivial! , Multiply powe

$$e^x \cdot e^{-x} = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)$$

$$= 1 + x - x + \frac{x^2}{2!} + \frac{x^2}{2!} - x^2 + \ldots$$

In general

$$e^x \cdot e^{-x} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!\,(n-k)!}\right) x^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}\right)\frac{x^n}{n!} = 1.$$

Since
Binomial theorem $\Rightarrow \;\; \forall\, n > 0$

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k}(-1)^{n-k}$$

2) If f is any other function s.t $f' = f$

Then prod / chain rule $\Rightarrow$

$$\frac{d}{dx} e^{-x} f = -e^{-x} f + e^{-x} f' = 0$$

$\Rightarrow \exists\, A \in \mathbb{R}$ s.t $e^{-x} f = A$.

1) $\Rightarrow$ $f(x) = A e^{x} \quad \forall\, x \in \mathbb{R} \Rightarrow f(0) = A$.

So $A e^{x}$ is the unique function satisfying both

3) Consider, for fixed $y \in \mathbb{R}$, the function

$$f_y(x) = e^{x+y}$$

$$f_y'(x) = e^{x+y} = f_y$$

$$f_y(0) = e^{y}$$

2) $\overset{(A = e^{y})}{\Longrightarrow}$ $f_y(x) = e^{y} \cdot e^{x} \Rightarrow e^{x+y} = e^{x} \cdot e^{y}$.

4) Sps $e^{x} = 0$. But then $e^{x} \cdot e^{-x} = 1$, contradicti

So $e^{x} \neq 0 \ \forall\, x$. e^{x} cont. $\Longrightarrow$ e^{x} cannot change sign. $e^{0} = 1 > 0 \Rightarrow e^{x} > 0 \ \forall\, x$

5) $\frac{d}{dx} e^{x} = e^{x} > 0 \Rightarrow e^{x}$ mon. increasing.

2) <u>Logarithm</u>: e^x strictly increasing $\Rightarrow$ e^x is one-one, and so has an inverse.

Note $e^x: \mathbb{R} \to (0,\infty)$ is surjective by intermediate value Th^m (since $e^x \xrightarrow{x\to\infty} \infty$, $e^x \xrightarrow{x\to-\infty} 0$).

<u>Def^n</u>: Define $\ln y : (0,\infty) \longrightarrow \mathbb{R}$ by

$$\ln y = x \iff e^x = y$$

<u>Th^m</u>: 1) $\ln 1 = 0$

2) $\ln y \xrightarrow{y\to\infty} \infty$

$\ln y \xrightarrow{y\to 0} -\infty$

3) $\ln(y_1 y_2) = \ln(y_1) + \ln(y_2)$

4) $\dfrac{d}{dy} \ln y = \dfrac{1}{y}$

5) $\ln x^p = p \ln x \quad \forall\, p, \cancel{x} \in \mathbb{R}$

<u>Application</u>: $\forall\, p \in \mathbb{R}\setminus\{0\}$,

$$\frac{d}{dx} x^p = \begin{cases} p x^{p-1} & x \neq 0 \\ 0 & p>1,\ x=0 \\ \text{Not diff} & x=0,\ p<1 \end{cases}$$

<u>Pf</u>: Let $y = x^p$, $x \neq 0$

$\ln y = p \ln x$

Diff w.r.t x: $\dfrac{d \ln y}{dx} = \dfrac{p}{x}$

Chain rule $\Rightarrow$ $\frac{y'}{y} = \frac{p}{x}$

$\Rightarrow$ $y' = p \cdot y/x = p \cdot x^p/x = p x^{p-1}$

At $x = 0$: $\frac{y(h) - y(0)}{h} = \frac{h^p}{h} = h^{p-1} \xrightarrow[p>1]{h \to 0} 0$

limit DNE if $p < 1$

3) Powers to arbitrary base

Thm: $x \in \mathbb{R}$, $a > 0$, Then

1) $a^x = e^{x \ln a}$

2) $\frac{d}{dx} a^x = a^x \ln a$

4) Sine / Cosine

Define $\sin(x), \cos(x) : \mathbb{R} \to \mathbb{R}$ by

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^{2n}$$