6. Differentiation

• The derivative

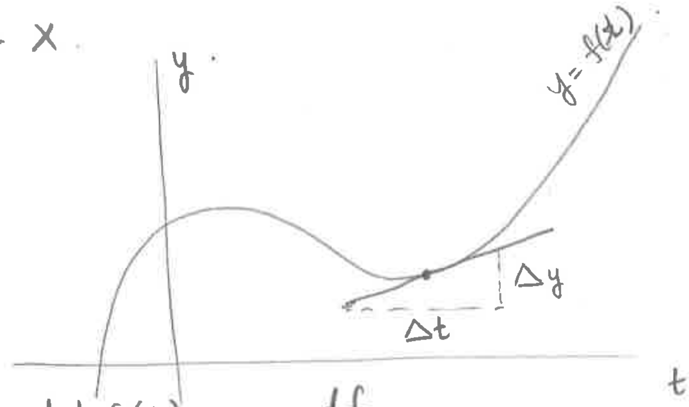
Def^n: Let $f:(a,b)\longrightarrow \mathbb{R}$ be defined. For any $x\in(a,b)$, we f is differentiable at x if

$$f'(x) := \lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists. We then denote by $f'(x)$, and call it the derivative of f at x. We say f is diff on (a,b) if it is diff at all $x\in(a,b)$.

Rk: 1) For diff we need x to be an interior point of (a,b).

2) Geometrically, $f'(x)$ calculates the slope of $y=f(t)$ at $t=x$.



3) Notation: $f'(x) = \left.\frac{d}{dt}\right|_{t=x} f(t) = \frac{df}{dx}$.

Th^m 5.1 Sps f,g are diff at $x\in(a,b)$. Then $f\pm g$ & fg are diff at x. f/g is also diff at x, if $g(x)\neq 0$. Moreover we have.

(a) $(f \pm g)' = f' \pm g'$

(b) $(cf)' = cf' \quad \forall c \in \mathbb{R}$.

(c) $(fg)' = f'g + g' \cdot f$.

(d) $(f/g)' = \dfrac{gf' - fg'}{g^2}$

<u>Pf</u>: Let us prove (c). Other parts are similar.

Let $h(t) = f(t)g(t)$

$$\begin{aligned} h(t) - h(x) &= f(t)g(t) - f(x)g(x) \\ &= f(t)g(t) - f(t)g(x) + f(t)g(x) - f(x)g(x) \\ &= f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)]. \end{aligned}$$

$$\Rightarrow \lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} f(t)\left[\frac{g(t) - g(x)}{t - x}\right] + g(x) \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

$$= f(x)g'(x) + g(x)f'(x).$$

<u>Examples</u>: 1) <u>Constant</u>: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = c$.

Then f' exists & $f'(x) = 0 \ \forall x$.

2) <u>Monomials</u>: $P_n: \mathbb{R} \to \mathbb{R}$

$x \to x^n$.

Claim: P_n is diff. on $\mathbb{R}$ $\forall$ x and

$$P_n'(x) = \begin{cases} n x^{n-1}, & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Pf: $n=0 \Rightarrow P_0(x) = 1 \Rightarrow P_0' = 0$. Sps $n \neq 0$.

Then
$$t^n - x^n = (t-x)(t^{n-1} + x t^{n-2} + \cdots + x^{n-1}).$$

$$\Rightarrow \quad \frac{t^n - x^n}{t-x} = t^{n-1} + x t^{n-2} + \cdots + x^{n-1} \xrightarrow{t \to x} n \cdot x^{n-1}.$$

$$\Rightarrow \quad \lim_{t \to x} \frac{P_n(t) - P_n(x)}{t-x} = n x^{n-1}.$$

3) Polynomials: Th^m 6.1 $\Rightarrow$ any

$$P : \mathbb{R} \to \mathbb{R}$$
$$P(t) = a_n t^n + \cdots + a_0.$$

is diff at any $x \in \mathbb{R}$

4) Absolute value: $f(x) = |x|$.

$$= \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

Clearly f is diff at every $x \neq 0$.

At $\underline{x=0}$: Let $\varphi(h) = \dfrac{f(h)-f(0)}{h} = \dfrac{|h|}{h}$

$$= \begin{cases} 1 & h \geq 0 \\ -1 & h < 0 \end{cases}$$

So $\lim_{h \to 0^+} \varphi(h) = 1 \neq -1 = \lim_{h \to 0^-} \varphi(h)$.

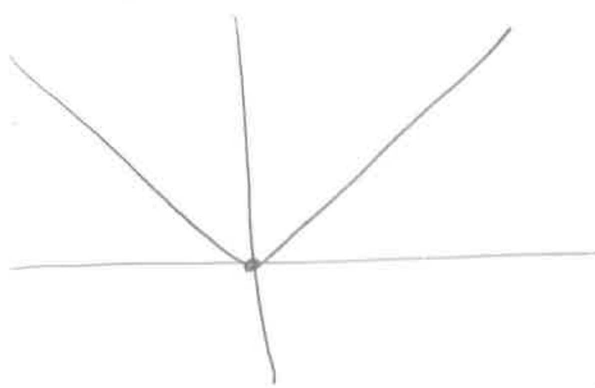
So f is <u>NOT</u> diff at $x=0$

<u>Rk</u>: 1) More generally for any $p \in \mathbb{R} \setminus \{0\}$

$$\frac{d}{dx} x^p = p x^{p-1}$$

We'll see a proof later

2) In example 4), the graph has a sharp



<u>corner</u> at $x=0$. This is prototypical of non-diff.

$\underline{Th^m}$ 5.2 $f : (a, b) \to \mathbb{R}$ diff at $x \implies f$ is cont at x.

$\underline{Pf}$: $f(t) - f(x) = \dfrac{f(t) - f(x)}{t - x} \cdot (t - x) \xrightarrow[\text{for lim}]{\text{prod. rule}} f'(x) \cdot 0 = 0$.

· Chain rule:

$\underline{Th^m}$ 5.3 Sps $f : [a, b] \to \mathbb{R}$ is cont. & diff at x.

Sps $g : I \to \mathbb{R}$, diff at $f(x)$. Then

$$h(t) = g \circ f(t) = g(f(t))$$

is diff at x &

$$h'(x) = g'(f(x)) \cdot f'(x).$$

$\underline{Pf}$:

$$\frac{h(t) - h(x)}{t - x} = \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \cdot \frac{f(t) - f(x)}{t - x}.$$

Now, $f(t) \to f(x)$ as $t \to x$ since f is cont at x.

So, as $t \to x$

$$\frac{h(t) - h(x)}{t - x} \to g'(f(x)) \cdot f'(x)$$

Done!

· Mean Value Theorems

$\underline{Def^n}$: (local max/min): Let X be metric space and $f : X \to \mathbb{R}$. We say $p \in X$ is a local max (min resp.) if $\exists\ \delta > 0$ s.t $\forall\ x \in B_\delta(p)$.

$$f(x) \leq f(p) \quad (\text{resp. } f(x) \geq f(p)).$$

<u>Th^m</u> 5.4 Let $f: [a,b] \to \mathbb{R}$. If f has a <u>local max</u> (or min.) at $p \in (a,b)$ and $f'(p)$ exist, then

$$f'(p) = 0.$$

<u>Pf</u>: Sps p is local max. (other case is identical) & $f'(p)$ exists. Then

$$f'(p) = \lim_{t \to p} \frac{f(t) - f(p)}{t - p}.$$

$\exists$ δ s.t $\forall$ $t \in (p-\delta, p+\delta)$, $f(t) \leq f(p)$.

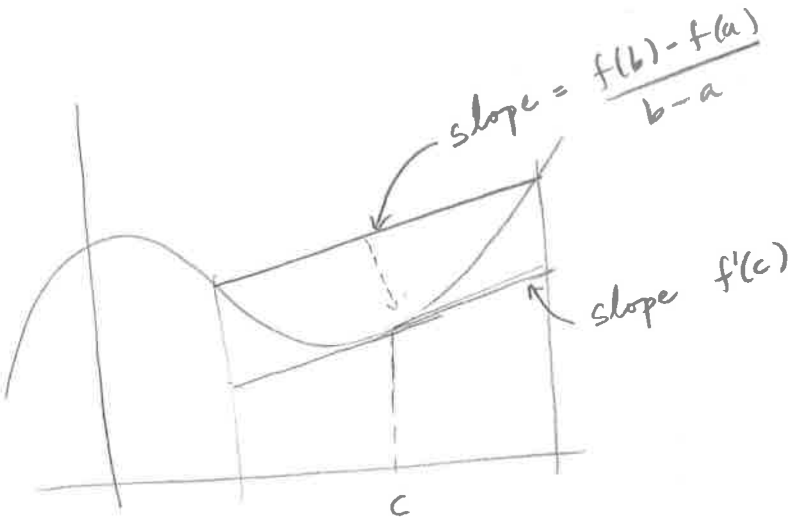
$$\Rightarrow \forall t \in (p-\delta, p+\delta),\ t \neq p, \quad \frac{f(t) - f(p)}{t - p} \begin{cases} \leq 0 & t > p \\ \geq 0 & t < p \end{cases}$$

So limit has to be zero!

$\Rightarrow$ $f'(p) = 0$.

<u>Th^m</u> (5.5) (Mean Value Theorem (MVT)) If $f: [a,b] \to \mathbb{R}$ cont and diff on (a,b), then $\exists$ $c \in (a,b)$ s.t

$$f(b) - f(a) = f'(c)(b-a).$$



<u>Pf</u>: Consider $h(t) = [f(b) - f(a)]t - [b-a]f(t)$; $a \le t \le b$.

Then h is diff. on (a,b) & cont. on $[a,b]$, $\boxed{h(a) = h(b)}$

To prove Th^m we need

<u>Aim</u>: $\exists$ $c \in (a,b)$. s.t $h'(c) = 0$.

<u>CASE 1</u> h is const $\Rightarrow$ $h'(t) = 0$ $\forall t$, so nothing to prove

<u>CASE 2</u>: h is not const. h cont. on $[a,b]$ $\Rightarrow$ $\exists$ max & min of h. Since $h(a) = h(b)$, either the max or the min is in the interior (a,b).

Say the max $c \in (a,b)$.

1st derivative test $\Rightarrow$ $f'(c) = 0$ Done!

<u>Rk</u>: Th^m not true if f is cont. only on (a,b).

Consider $f(t) = \begin{cases} t & 0 \le t < 1 \\ 2 & t = 1 \end{cases}$

Then $\dfrac{f(1) - f(0)}{1 - 0} = 2$, but $f'(t) = 1 < 2$.

<u>Cor 5.6</u> Sps f is diff on (a, b).

(a) $f'(x) \geq 0$ (resp ≤ 0) on $(a,b) \Rightarrow f$ is increasing (resp. decreasing) on (a,b)

(b) $f'(x) = 0$ on $(a,b) \Longrightarrow f = \text{const}$.

<u>Pf</u>: (b).

For any $s, t \in (a,b)$, f is cont. on $[s,t]$.
f is diff on (s,t).

$\cancel{\exists}$ MVT $\Rightarrow \exists\ c$ s.t.

$$\frac{f(t) - f(s)}{t - s} = f'(c) = 0$$

$\Rightarrow f(t) = f(s) \quad \forall\ s, t \in (a,b)$

$\Rightarrow f = \text{const}$

(a) Pf is similar.

• <u>Taylor's Theorem</u>: We denote higher order derivative by $f', f'', f''', f^{(4)}, \ldots, f^{(n)}$ etc.

Note that if $f^{(k)}$ exists, $f', \ldots, f^{(k-1)}$ are also cont

$\underline{\text{Def}^n}$: We say f in C^k on $[a,b]$ if $f', \dots, f^{(k)}$ exist on (a,b) and are cont. on $[a,b]$

We say f is smooth or $C^\infty[a,b]$ if f in $C^k[a,b]$ $\forall$ k.

$\underline{\text{e.g}}$ Polynomials.

$\underline{\text{Th}^m}$ (Taylor's theorem) Sps f in $C^n[a,b]$ s.t $f^{(n+1)}$ also exist on (a,b). For any p, q in (a,b), $\exists$ $t_{p,q}$ between q & p s.t

$$f(q) = \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(q-p)^k + \frac{f^{(n+1)}(t_{p,q})}{(n+1)!}(q-p)^{n+1}$$

Denoting $f(q) = T_n(q) + R_n(q)$, if $M := \sup_{t \in [a,b]} |f^{(n+1)}(t)|$ exist, then

$$|R_n(q)| < \frac{M}{(n+1)!}|q-p|^{n+1}$$

$\underline{\text{Pf}}$: Let $T_n(q) = \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(q-p)^k$

For $p \neq q$, define

$$M = \frac{f(q) - T_n(q)}{(q-p)^{n+1}}$$

$\underline{\text{Aim}}$: $M = f^{(n+1)}(t_{p,q}) \cancel{(n+1)!}$ for some $t_{p,q}$ between p, q.

Consider

$$g(t) = f(t) - T_n(t) - M(t-p)^{n+1}$$

Clearly $\quad g^{(n+1)}(t) = f^{(n+1)}(t) - M(n+1)!$

Since $\quad T^{(k)}(p) = f^{(k)}(p)$ for $k = 0, 1, \dots, n$

$$g^{(k)}(p) = 0 \quad \text{for} \quad k = 0, 1, \dots, n$$

Moreover $\quad g(q) = 0$ by choice of M

So MVT $\Rightarrow \exists\, t_1$ between p, q s.t $g'(t_1) = 0$.

Since $g'(p) = 0$, $\exists\, t_2$ between p, t_1 (and hence also between p & q) s.t $g''(t_2) = 0$

After n steps, $\exists\, t_n$ between p, q s.t $g^{(n)}(t_n) = 0$

Again $g^{(n)}(p) = 0 \Longrightarrow \exists\, t_{p,q}$ between p, q s.t

$$g^{(n+1)}(t_{p,q}) = 0 \iff M = \frac{f^{(n+1)}(t_{p,q})}{(n+1)!} \qquad \text{Done!}$$

• ~~<u>Extremum Values / Convexity</u>~~

~~<u>Th</u>^{m} (Second derivative test). Let f~~

- Application of derivative

1) Extremum value problems

Def^n: A point p is a local extrema for $f : X \to \mathbb{R}$ if p is a local max/min. It is called a critical point if either f is not diff at p or $f'(p) = 0$

Th^m: Let $f : [a,b] \longrightarrow \mathbb{R}$ be cont.

1) If $p \in (a,b)$ is a local extrema, then p is a critical point.
2) If p is critical point s.t. $f'(p) = f''(p) = \dots = f^{(n)}(p) = 0$ & $f^{(n+1)}(p) \neq 0$. Additionally sps $f^{(n+1)}$ cont. near p. Then.
   (a) n is odd, $f^{(n+1)}(p) > 0 \Rightarrow f$ has local min at p
   (b) n is odd, $f^{(n+1)}(p) < 0 \Rightarrow f$ has local max at p.
   (c) n is even $\Longrightarrow f$ has neither. p is then called an inflection point.

Pf: 1) Already prove (Th^m 5.)

2). Sps $f \in C^{n+1}$ near p. and sps $f^{(n+1)}(p) > 0$. Since $f^{(n+1)}$ is cont. near p, $\exists\ \delta$ s.t. $f^{(n+1)}(t) > 0$ $\forall\ t \in (p-\delta, p+\delta)$.

Claim: If n is odd, then $\forall\, q \in (p-\delta, p+\delta)$,

$$f(q) > f(p)$$

Pf: Taylor's theorem $\Rightarrow \exists\ t_{p,q}$ between p & q

s.t.

$$f(q) = \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(q-p)^k + \frac{f^{(n+1)}(t_{p,q})}{(n+1)!}(q-p)^{n+1}$$

$$f^{(k)}(p) = 0 \quad \text{for} \quad k = 1, 2, \cdots, n,$$

$$\Rightarrow \quad f(q) = f(p) + \frac{f^{(n+1)}(t_{p,q})}{(n+1)!}(q-p)^{n+1}$$

Note that $t_{p,q} \in (p-\delta, p+\delta) \Rightarrow f^{(n+1)}(t_{p,q}) > 0$

If $q > p \Rightarrow q - p > 0 \Rightarrow f(q) > f(p)$

If $q < p \Rightarrow (q-p)^{n+1} > 0$ since $n+1$ is even

$$\Rightarrow f(q) > f(p)$$

So claim is proved

(b) & (c) are proved similarly

<u>Cor</u>: A cont function $f: [a,b] \to \mathbb{R}$ has an extremum at p. Then one of the foll holds

(1) $p = a$ or b.

(2) f is not diff at p.

(3) $f'(p) = 0$

<u>Examples</u>: 1) $f(x) = x^2$ on $[-1, 1]$.

$f'(x) = 2x$, $f'(0) = 0$.

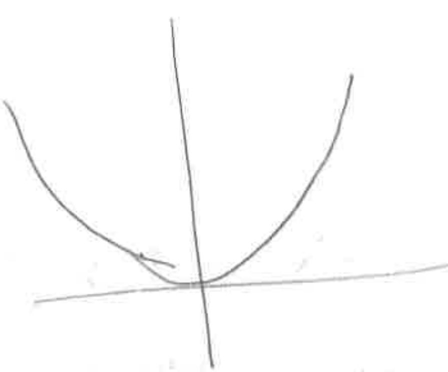
So $x = 0$ only critical point

$f''(x) = 2 > 0 \Rightarrow f$ has local min at $x=0$.

Possible extremas $= -1, 0, 1$, $f(-1) = f(1) = 1$, $f(0) = 0$.

So Max at $x = \pm 1$

Min at $x = 0$



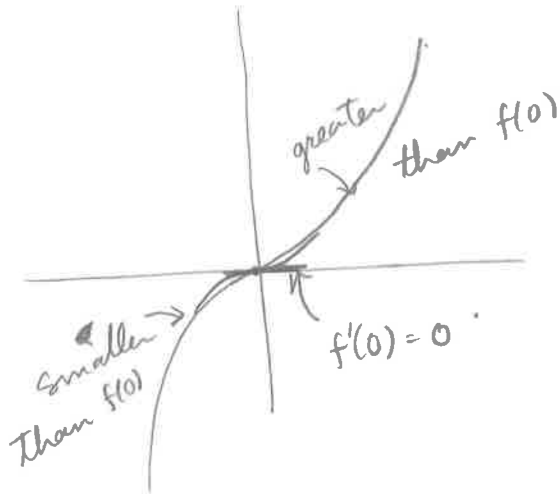
2) $f(x) = x^3$, $f'(x) = 3x^2$. So $x = 0$ critical point
on $[-1, 1]$.

$f''(x) = 6x$, $f''(0) = 0$

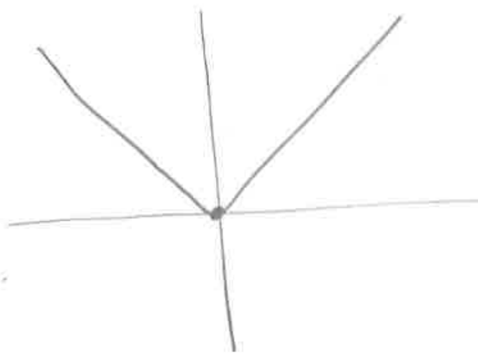
$f'''(x) = 6 \neq 0$.

So $n = 2$ (even) in the last Th^m

$\Rightarrow x = 0$ is an inflection point



3) $f(x) = |x|$



$x = 0$ critical point & <u>local min</u>.

2) <u>L'Hopitals rule</u>

<u>Th</u>^{m}: Sps f & g are real diff on (a,b), $g'(x) \neq 0$

Sps $c \in (a,b)$

(i) $f(x), g(x) \xrightarrow{x \to a} 0$ OR (ii) $g(x) \to \infty$ as $x \to a$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A$$

2) L' Hopitals rule

Th^m: Sps f, g are in $C^n(a,b)$

(a) Sps $c \in (a,b)$ s.t $f(c) = f'(c) = \cdots = f^{(n-1)}(c) = 0$.
& $g(c) = \cdots = g^{(n-1)}(c) = 0$ but $g^{(n)}(c) \neq 0$ on (a,b).

Then
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}$$

(b) Sps $g(x) \to \infty$ as $x \to a^+$ & $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = A$

$$\Longrightarrow \quad \lim_{x \to a^+} \frac{f(x)}{g(x)} = A$$

Pf (a) First Sps $n=1$ i.e $f(c) = g(c) = 0$, $g'(c) \neq 0$

$$\frac{f(x)}{g(x)} = \left(\frac{f(x) - f(c)}{x - c}\right) \cdot \left(\frac{x-c}{g(x) - g(c)}\right) = \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}}$$

$$\xrightarrow{x \to c} \frac{f'(c)}{g'(c)} \quad \text{since } g'(c) \neq 0$$

If $n > 1$, we proceed by induction Sps Th^m proved for n. Sps $f, g \in C^{n+1}$ s.t $f(c) = \cdots = f^{(n)}(c) =$
$g(c) = \cdots = g^{(n)}(c) = 0$

but $g^{(n+1)}(c) \neq 0$. Then $F = f'$, $G = g'$ satisfy ; $F, G \in C^n$
$F(c) = \cdots = F^{(n-1)}(c) = G(c) = \cdots G^{(n-1)}(c) = 0$.

And $G^{(n)}(c) \neq 0$

$\Rightarrow$ $\xRightarrow[\text{hyp}]{\text{Induction}}$ $$\lim_{x\to c} \frac{F(x)}{G(x)} = \frac{F^{(n)}(c)}{G^{(n)}(c)} = \frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}$$

i.e $$\lim_{x\to c} \frac{f'(x)}{g'(x)} = \frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}$$

Claim: $$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}$$

Pf: let $f^{(n+1)}(c) / g^{(n+1)}(c) = A$

Given $\varepsilon > 0$, $\exists$ δ s.t

$$|x-c| < \delta \implies A - \varepsilon < \frac{f'(x)}{g'(x)} < A + \varepsilon \qquad (*)$$

$x \neq c$

Sub claim: For any $p \in (a,b)$. $\exists$ x_0 between c and p

s.t $$f(p) \cdot g'(x_0) = g(p) f'(x_0)$$

Pf: Consider $$h(x) = f(x)\, g(p) - g(x)\, f(p)$$

$h(c) = 0$, $h(p) = 0$

MVT $\Rightarrow$ $\exists$ x_0 between p, c s.t $h'(x_0) = 0$

$$h'(x_0) = f'(x_0)\, g(p) - g'(x_0) f(p) = 0$$ Done!

Now. sps $|p-c|<\delta$. $\exists$ x_0 between p, c s.t

$$f(p)\, g'(x_0) = g(p)\, f'(x_0)$$

By Taylor's theorem,

$$g(p) = \sum_{k=0}^{n-1} \frac{g^{(k)}(c)}{k!}(p-c)^k + \frac{g^{(n)}(t_{p,c})}{n!}(p-c)^n$$

$$= \frac{g^{(n)}(t_{p,c})}{n!}(p-c)^n$$

$g^{(n)}(c) \neq 0$ & $g^{n}(t)$ cont $\Rightarrow$ if δ small, can make sure $g^{(n)}(t_{p,c}) \neq 0$ $\Rightarrow$ $g(p) \neq 0$ $\forall$ $|p-c|<\delta$, $p\neq c$. if δ small.

Similarly $g'(x_0) \neq 0$ if δ small enough.

$$\Rightarrow \quad \frac{f(p)}{g(p)} = \frac{f'(x_0)}{g'(x_0)}$$

$$(*) \Rightarrow \quad A-\varepsilon < \frac{f(p)}{g(p)} < A+\varepsilon$$

So given $\varepsilon > 0$, $\exists$ δ s.t $\left|\frac{f(p)}{g(p)} - A\right| < \varepsilon$.

$\forall$ p s.t $|p-c|<\delta$.

$\Rightarrow$ $\lim\limits_{p\to c} \frac{f(x)}{g(x)} = A$. This proves Claim & completes inductive Step.

(b) See Rudin (pg 110)