Ascoli - Arzela & Stone - Weierstrass

• Ascoli - Arzela

Recall that a subset $K \subset X$ of a metric sp is **compact** if and only if for any seq^n $\{x_n\}$ in K there exist a sub-sequence $\{x_{n_k}\}$ & $p \in K$ s.t

$$\lim_{k \to \infty} x_{n_k} = p.$$

Ques: Given a seq $\{f_n\}$ of cont. real valued function on $E \subset X$, when is there a sub-sequence that converges uniformly? What about pointwise?

Example: Let $f_n(x) = x^n$ on $[0,1]$.

Claim: No subsequence f_{n_k} converges uniformly on $[0,1]$

Pf: If there is such a sub-seq f_{n_k}, then

$$f_{n_k} \xrightarrow{u.c} f = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$$

But f_{n_k} is cont. for each k, while f is not contradicting the theorem on uniform conv. & cont.

We need to introduce 2 new concepts. Let $X = \mathbb{R}$.

<u>Def^n</u> A <u>family of</u> function $\mathcal{F}$ defined on a subset $E \subset \mathbb{R}$ is called <u>uniformly</u> bounded if $\exists\ M > 0$ s.t

$$|f(x)| \leq M \quad \forall f \in \mathcal{F},\ \forall x \in E.$$

It is said to be <u>pointwise</u> bounded if for each $x \in E$, $\exists\ M(x)$ s.t

$$|f(x)| < M(x) \quad \forall f \in \mathcal{F}.$$

<u>Example</u> 1) Consider $f_n(x) = x^n$ on $[0,1]$. Then

$$|f_n(x)| \leq 1 \quad \forall n, \quad \forall x \in [0,1].$$

So $\{f_n\}$ is uniformly bounded on $[0,1]$.

2) If $f_n(x) = nx$ on $[0,1]$. Then for each n, $f_n(x)$ is bounded on $[0,1]$ but is <u>NOT</u> pointwise bounded if $x \neq 0$. since for any $x \neq 0$ $f_n(x) \to +\infty$.

3) Let $f_n(x) = n^2 x^{n+1}(1-x)$ on $[0,1]$. Clearly for each $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = 0$ So $\{f_n\}$ is pointwise bounded, but

$$f_n\left(\frac{n}{n+1}\right) = n^2\left(\frac{n}{n+1}\right)^{n+1} \cdot \frac{1}{n} = n\left(1 - \frac{1}{n+1}\right)^{n+1}$$

$$\approx n/e \longrightarrow +\infty$$

So there is no uniform bound that will work for all x

$\underline{\text{Th}^m}$: Let $f_n : E \to \mathbb{R}$ s.t each f_n is bounded. That is, $\forall n$, $\exists$ M_n s.t

$$|f_n(x)| < M_n \text{ on } E$$

If $f_n \xrightarrow{u.c} f$ on E, then $\{f_n\}$ is uniformly bounded.

$\underline{\text{Pf}}$: $\exists$ N s.t $\forall n > N$, $\forall x \in E$

$$|f_n(x) - f(x)| < 1. \qquad (f_n \xrightarrow{u.c} f) \quad (*)$$

In particular

$$f_N(x) - 1 < f(x) < f_N(x) + 1$$

$$\Rightarrow \quad -M_N - 1 < f(x) < M_N + 1$$

$$\Rightarrow \quad |f(x)| < M_N + 1 \quad \forall x \in E$$

So f is bounded.

Again by $(*)$, $\forall n \geq N$, $\forall x \in E$

$$|f_n(x)| < 1 + |f(x)|$$
$$< M_N + 2.$$

Let $M = \max(M_1, \dots, M_{N-1}, M_N + 2)$

If $n < N$, then $|f_n(x)| < M_n \leq M \quad \forall x \in E$

$n \geq N$, then $|f_n(x)| < M_N + 2 \leq M \quad \forall x \in E$

$$\Rightarrow \quad |f_n(x)| \leq M \quad \forall n, \forall x \in E$$

Def^n: A family of functions $\mathcal{F}$ on $E \subset X$ is called <u>equicontinuous</u> if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon.$$

$\forall\ f \in \mathcal{F}$, $\forall\ x \in E$.

<u>Rk</u>: In particular each $f \in \mathcal{F}$ is <u>uniformly</u> cont.

<u>Th^m</u>: If X is compact, and $f_n : X \to \mathbb{R}$ continuous s.t. $f_n \xrightarrow{u.c} f$ on X. Then $\{f_n\}$ is equicontinuous on X.

<u>Pf</u>: Given $\varepsilon > 0$, $\exists\ N$ s.t. $\forall\ x \in X$

$$|f_n(x) - f_N(x)| < \varepsilon/3$$

For each $i = 1, \dots, N$, $\exists\ \delta_i$ s.t.

$$d(x,y) < \delta \implies |f_i(x) - f_i(y)| < \varepsilon/3$$

This is because X compact & so $\{f_n\}$ are uniformly cont. for each n.

Let $\delta = \min(\delta_1, \dots, \delta_N)$. Sps $|x - y| < \delta$.

Then for $n \leq N$, by our choice of δ

$$|f_n(x) - f_n(y)| < \varepsilon/3 < \varepsilon.$$

If $n > N$ then

$$|f_n(x) - f_n(y)| < |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

$$< \frac{\varepsilon}{3} \cdot 3 = \varepsilon.$$

<u>Th^m</u> (Ascoli - Arzela). Let $K \subset \mathbb{R}^n$ be compact and $\{f_n\}$ is a seq^n of <u>point wise</u> bounded and equicont. functions on K. Then

1) $\{f_n\}$ is uniformly bounded.

2) $\exists$ sub-seq^n $\{f_{n_k}\}$ which converge uniformly.

• <u>Compactness in Function Spaces</u>. $K \subset \mathbb{R}^n$ compact

let

$$C^0(K) = \{ f : K \to \mathbb{R} \mid f \text{ cont.} \}$$

$$d(f, g) = \sup_{\vec{x} \in K} |f(\vec{x}) - g(\vec{x})|, \quad f, g \in C^0(K).$$

<u>FACT</u>: ($n=1$, this is a homework problem) $(C^0(K), d)$ forms a <u>complete</u> metric space. $f_n \xrightarrow{d} f \iff f_n \xrightarrow{u.c} f$ on K.

$\underline{\text{Th}^m}$ (Ascoli - Arzela ver - 2). A subset $\mathcal{F} \subset C^0(K)$ is compact $\Longleftrightarrow$ $\mathcal{F}$ is closed, bounded & equi cont.

Pf: $\underline{\Longleftarrow}$ Sps $\{f_n\}$ any seq in $\mathcal{F}$. $\{f_n\}$ bounded, equicont. $\xRightarrow{A-A}$ $\exists$ sub-seq f_{n_k} s.t $\{f_{n_k}\}$ conv. uniformly to f on K. $\Rightarrow$ $f_{n_k} \xrightarrow{d} f$.

$\mathcal{F}$ closed $\Rightarrow$ $f \in \mathcal{F}$. So every infinite seq in $\mathcal{F}$ has a l.p in $\mathcal{F}$ $\Rightarrow$ $\mathcal{F}$ is compact

$\Longrightarrow$ $\mathcal{F}$ compact $\Rightarrow$ $\mathcal{F}$ is closed & bounded. Sps $\mathcal{F}$ is not equi cont. Then $\exists$ ε, and seq f_n in $\mathcal{F}$ & x_n, y_n in K s.t

$$|x_n - y_n| < 1/n \quad , \quad \underline{\underline{\text{but}}} \quad |f_n(x_n) - f_n(y_n)| > \varepsilon .$$

But $\mathcal{F}$ comp. $\Rightarrow$ a sub-seq (which we assume is f_n itself) conv. uniformly to f

$\exists$ N s.t $\forall$ $n \geq N$, $\forall$ $x \in K$

$$|f(x) - f_n(x)| < \varepsilon/3$$

Also f cont, since f_n s are cont. & $f_n \xrightarrow{u.c} f$

K compact $\Rightarrow$ f uniformly cont. So $\exists$ δ s.t $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/3$.

Choose $n > N$, $n > 1/\delta$. Then $|x_n - y_n| < 1/n < \delta$.

So

$$|f_n(x_n) - f_n(y_n)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - f_n(y_n)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Contradiction. So $\mathcal{F}$ is also equicont.

• <u>Proof of Ascoli - Arzela</u>

<u>Lemma</u>: Let E be countable. If $\{f_n\}$ is a pointwise bounded seq^n of functions, then there is sub-seq $\{f_{n_k}\}$ s.t $\{f_{n_k}(x)\}$ converges (pointwise) for all $x \in E$.

<u>Pf</u>: Let $\{x_1, x_2, \ldots\}$ be points in E. Since $\{f_n(x_1)\}$ is <u>bounded</u> $\Longrightarrow \exists$ <u>sub-seq</u> $\{f_{1,k}\}$ s.t $f_{1,k}(x_1)$ converges as $k \to \infty$

<u>REASON</u>: Any bounded seq in $\mathbb{R}$ is contained in a compact interval.

Now, consider $\{f_{1,k}(x_2)\}$. This is again bounded. So there exists sub-seq $\{f_{2,k}\}$ s.t $\{f_{2,k}(x_2)\}$ converges as $k \to \infty$.

More generally, construct sequence $S_1, S_2, \ldots$

$$S_1: f_{1,1}, f_{1,2}, f_{1,3}, f_{1,4} \ldots$$

$$S_2: f_{2,1}, f_{2,2}, f_{2,3} \ldots$$

$$S_3: f_{3,1}, f_{3,2}, f_{3,3} \ldots$$

s.t

1) S_n is a sub-seq of S_{n-1}. S_1 sub-seq of $\{f_n\}$.

2) $\{f_{n,k}(x_n)\}$ converges as $k \to \infty$.

3) functions appear in same order in S_n as they appear in $\{f_n\}$

Now, consider the "diagonal"

$$S: f_{1,1}, f_{2,2}, f_{3,3}, \ldots f_{n,n}, \ldots$$

Claim: $\{f_{n,n}(x_m)\}$ converges $\forall m$.

Pf: Given x_m, note $\{f_{n,n}(x_m)\}_{n \geq m}$ is a sub-seq of S_m at x_m: $f_{m,k}(x_m)$ which converges

$\Rightarrow \{f_{n,n}(x_m)\}$ converges

Proof of Ascoli - Arzela

(a) $\{f_n\}$ equicont. So $\exists\ \delta > 0$ s.t $\forall n$,

$$|x-y| < \delta \implies |f_n(x) - f_n(y)| < 1.$$

Consider covering $K \in \bigcup_{p \in K} B_\delta(p)$.

K compact $\implies \exists\ p_1, \cdots, p_\ell$ s.t $K \subset \bigcup_{i=1}^{\ell} B_\delta(p_i)$.

i.e any $x \in K$, $\exists\ p_k$ s.t $|x - p_k| < \delta$.

$\{f_n\}$ pointwise bounded $\implies \forall\ p_k\ \exists\ M_k$ s.t

$$|f_n(p_k)| < M_k \quad \forall n.$$

Let $M = \max(M_1, \cdots, M_\ell) + 1$.

<u>Claim</u>: $|f_n(x)| < M \ \forall x \in K, \forall n$.

<u>Pf</u>: $x \in K \implies \exists\ p_k$ s.t $|x - p_k| < \delta$

$$\implies |f_n(x)| \le |f_n(x) - f_n(p_k)| + |f_n(p_k)|$$
$$< 1 + M_k < M$$

So $\{f_n\}$ uniformly bounded.

(b) Let $E \subset K$ be a <u>countable</u> <u>dense</u> subset.
One can take E to be the set of points with <u>rational coordinates</u>

Lemma $\Rightarrow$ $\exists$ subseq f_{n_j} s.t $f_{n_j}(x)$ conv. $\forall$ $x \in E$.

Let $g_j = f_{n_j}$, $E = \{p_1, \dots\}$.

<u>Claim</u>: g_j conv. <u>uniformly</u> on K.

<u>Pf</u>: Given ε, pick δ s.t $\forall$ n.

$$|x - y| < \delta \implies |g_j(x) - g_j(y)| < \varepsilon/3 \quad (*) \text{ (equicont.)}$$

Since $E \subset K$ is dense. $K \subset \bigcup_{i=1}^{\infty} B_\delta(p_i)$.

K comp. $\Longrightarrow$ $\exists$ $p_1, \dots, p_m$ s.t

$$K \subset B_\delta(p_1) \cup \dots \cup B_\delta(p_m)$$

$g_j(p_k)$ conv for $k = 1, \dots, m$ So $\exists$ N s.t $\forall$ $i, j \geq N$

$$|g_j(p_k) - g_i(p_k)| < \varepsilon/3 \ \forall \ k = 1, \dots, m.$$

(Cauchy criteria for pointwise conv.)

If $x \in K$, $\exists$ $k \in \{1, 2, \dots, m\}$ s.t $|x - p_k| < \delta$.

$(*) \Rightarrow$ $|g_j(x) - g_j(p_k)| < \varepsilon/3$. $\forall$ j.

If $i, j > N$ then

$$|g_j(x) - g_i(x)| < |g_j(x) - g_j(p_k)| + |g_j(p_k) - g_i(p_k)| + |g_i(p_k) - g_i(x)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Since N independent of x, $\{g_j\}$ is uniformly Cauchy $\Rightarrow$ $\{g_f\}$ conv. uniformly.

· Stone - Weierstrass

Th^m: If f is a cont. function on $[a,b]$, $\exists$ seq^n of polynomials P_n s.t $P_n \xrightarrow{u.c} f$ on $[a,b]$.

Pf: Step 1: One can assume $[a,b] = [0,1]$ by translation and scaling. We may also assume $f(0) = f(1) = 0$. If, not, then

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad , \quad 0 \leq x \leq 1$$

satisfies $g(0) = g(1) = 0$. Note $g(x) - f(x)$ is a polynomial. If $P_n \xrightarrow{u.c} g$, then $Q_n = P_n - f(x) + g(x)$ is a seq^n of polynomials s.t $Q_n \xrightarrow{u.c} f$.

From now, assume $f(0) = f(1) = 0$. Extend f to $\mathbb{R}$ by letting $f(x) = 0 \; \forall \; x \in (-\infty, 0] \cup [1, \infty)$.

Step 2: Consider

$$Q_n(x) = c_n (1 - x^2)^n$$

where c_n is chosen so that

$$\int_{-1}^{1} Q_n(t)\,dt = 1$$

Claim: $c_n < \sqrt{n}$

Pf: First note Bernoulli's inequality

$$(1-x^2)^n \geq 1 - nx^2 \qquad \forall\, x \in (0,1).$$

To prove this, let $h(x) = (1-x^2)^n - 1 + nx^2$.

$h(0) = 0$. $\quad h' = n(1-x^2)^{n-1} \cdot (-2x) + 2nx$.

$$= 2nx\left[1 - (1-x^2)^{n-1}\right] \geq 0 \quad \text{if } x \in (0,1)$$

So h increases in $(0,1)$ $\Rightarrow$ $h(x) \geq 0$ in $(0,1)$.

Now,
$$\int_{-1}^{1} (1-x^2)^n\,dx = 2\int_0^1 (1-x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

But $\quad c_n \displaystyle\int_{-1}^{1} (1-x^2)^n\,dx = 1 \Rightarrow c_n < \sqrt{n}$.

Step 3: For any $1 > \delta > 0$, $x \in [\delta, 1]$

$$Q_n(x) < \sqrt{n}\,(1-x^2)^n < \sqrt{n}\,(1-\delta^2)^n$$

$\Rightarrow$ On $[\delta, 1]$, $Q_n(x) \xrightarrow{u.c.} 0$. since

$$\lim_{n\to\infty} \sqrt{n}\,(1-\delta^2)^n = 0 \quad \text{if } 1 > \delta > 0$$

Step 4: Set

$$P_n(x) = \int_0^1 f(t)\, Q_n(x-t)\, dt.$$

Since Q_n is a polynomial, and we integrate out $f(t)$ $\Longrightarrow$ $P_n(x)$ is a poly.

Claim: $P_n \xrightarrow{u.c} f$ on $[0,1]$.

Pf: Change variables. $s = x - t$.

$$P_n(x) = -\int_x^{x-1} f(x-s)\, Q_n(s)\, ds$$

$$= \int_{x-1}^{x} f(x-s)\, Q_n(s)\, ds. \quad , \quad x \in [0,1]$$

For any x, s varies from $x-1$ to x, then $x-s$ varies from 1 to 0. Since $f \equiv 0$ outside $[0,1]$, if we ~~[illegible]~~ let s vary from $[-1,1]$, then integral remains unchanged.

So $$P_n(x) = \int_{-1}^{1} f(x-s)\, Q_n(s)\, ds.$$

Let $\varepsilon > 0$. $\exists$ δ s.t

$|x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon/2$.

Can do since f is uniformly cont on $[0,1]$.

Let $M = \sup_{t \in [0,1]} |f(t)|$. Since $Q_n(s) \geq 0$

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x-s)\, Q_n(s)\,ds - \int_{-1}^{1} f(x)\, Q_n(s)\,ds \right|$$

$$\leq \int_{-1}^{1} |f(x-s) - f(x)|\, Q_n(s)\,ds$$

$$= \left(\int_{-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{1} \right) |f(x-s) - f(x)|\, Q_n(s)\,ds$$

$$\leq 2M \int_{-1}^{-\delta} Q_n(s)\,ds + \int_{-\delta}^{\delta} |f(x-s) - f(x)|\, Q_n(s)\,ds$$

$$+ 2M \int_{\delta}^{1} Q_n(s)\,ds$$

Note $Q_n(s) \leq \sqrt{n}(1-\delta^2)^n$ on $[-1,-\delta] \cup [\delta, 1]$.

$$\Rightarrow |P_n(x) - f(x)| \leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2} \underbrace{\int_{-\delta}^{\delta} Q_n(s)\,ds}_{\leq 1}$$

$$\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2}$$

Choose N s.t. $\forall\, n > N$, $\sqrt{n}(1-\delta^2)^n \leq \varepsilon/8M$.

$\Rightarrow \forall n \geq N$, $|P_n(x) - f(x)| \leq \varepsilon \quad \forall\, x \in [0,1]$