Assignment-8

(Due 04/05)

1. Suppose f is a bounded real valued function on $[a, b]$ such that $f^2 \in \mathcal{R}[a, b]$. Does it follow that $f \in \mathcal{R}[a, b]$? Does the answer change if we assume $f^3 \in \mathcal{R}[a, b]$? Either give a proof or provide a counter example in each of the two cases.

2. (a) Let $f \in \mathcal{R}[a, b]$ and $\{p_1, \cdots p_n\}$ be a finite collection of points in $[a, b]$. Let $g : [a.b] \to \mathbb{R}$ be a bounded function such that
$$f(t) = g(t),$$
for all $t \in [a, b] \setminus \{p_1, \cdots, p_n\}$. Show that $g \in \mathcal{R}[a, b]$ and that
$$\int_a^b g(t)\, dt = \int_a^b f(t)\, dt.$$
Hint. Do it for one point at a time.

   (b) Is the conclusion true, if we instead have a countable collection of points $\{p_n\}_{n=1}^{\infty}$? **Hint.** What is the most basic non-integrable function that you know?

3. Give an example of functions $f \in \mathcal{R}[a, b]$ and $\varphi \in \mathcal{R}[m, M]$ where $m < f < M$, but such that $\varphi \circ f$ is not integrable. This shows that continuity of φ is necessary in the theorem proved in class on integrability of composition of functions.

   Hint. Construct f and φ such that the composition (defined maybe on $[0, 1]$) is one on rationals and 0 on irrationals. For φ you might want to consider the function in Prob. 5 in Assignment-6. Why is that function integrable on $[0, 1]$? And what should then be your choice for f?