ASSIGNMENT-3

This assignment has 6 problems. The final problem is not to be written up.

(1) For any two sequences $\{a_n\}$ and $\{b_n\}$, show that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the right side is not of the form $\infty - \infty$ or $-\infty + \infty$. Give an example of strict inequality.

(2) Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for the following sequences.
   (a) $a_n = \cos n$
   (b) $a_n = n \sin \frac{n\pi}{3}$.

(3) If $s_1 = \sqrt{2}$, and for $n = 1, 2, \cdots$, we define recursively

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}},$$

prove that $\{s_n\}$ converges. **Hint.** Show that s_n is monotonic, and that $s_n < 2$ for all n.

(4) Show that the metric space $(l^\infty(\mathbb{C}), d)$ from the previous assignment is complete. This shows (along with the problem 3 in the previous assignment) that completeness does not guarantee compactness of closed and bounded sets.

(5) The aim of this exercise is to generalize Bolzano-Weierstrass theorem. A metric space X is called *totally bounded* if for any $r > 0$, there is a finite covering of X by balls of radius r.
   (a) Show that a set in $\mathbb{R}^n$ (considered as a metric subspace) is totally bounded if and only if it is bounded.
   (b) A metric space is compact if and only if it is complete and totally bounded.
   Hint. Compactness implies completeness and total boundedness is easy. For the converse, proceed by contradiction, and use an argument similar to the proof of compactness of k-cells.

(6) **(not to be submitted)** The aim of this exercise is to give another construction of the real numbers. Let (X, d) be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$ and $\{q_n\}$ in X *equivalent* (denoted by $\{p_n\} \sim \{q_n\}$) , if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Show that this is an equivalence relation. That is show that the relation is **symmetric** (i.e $\{p_n\} \sim \{q_n\} \implies \{q_n\} \sim \{p_n\}$) , **reflexive** (ie. $\{p_n\} \sim \{p_n\}$), and **transitive** (ie. $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$ implies $\{p_n\} \sim \{r_n\}$).

(b) Let $\{p_n\}$ and $\{q_n\}$ be two Cauchy sequences in X. Show that the sequence $\{d(p_n, q_n)\}$ converges. **Hint.** Show that the sequence (which consists of real numbers) is Cauchy.

(c) Let X^* be the set of all equivalence classes of Cauchy sequences. If $P \in X^*$ and $Q \in X^*$ with $\{p_n\} \in P$ and $\{q_n\} \in Q$, then define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n).$$

Show that this number is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced with equivalent Cauchy sequences.

(d) Show that X^* with Δ is a complete metric space.

(e) For each $p \in X$, there is a Cauchy sequence all of whose terms are p. Let P_p be the element in X^* which contains this sequence. This defines a mapping

$$\Phi : X \to X^*.$$

Show that this mapping is distance preserving. That is, $\Delta(P_p, P_q) = d(p, q)$.

(f) Show that $\Phi(X)$ is dense in X^*, and that $\Phi(X) = X^*$ if and only if X is complete.

Remark. In this sense, we may regard X as *embedded* in X^*, and X^* is called the completion of X. We can then define $\mathbb{R}$ as the completion of the rationals $\mathbb{Q}$.