ASSIGNMENT-2
(DUE 02/08)

(1) Are the closures and interiors of connected sets always connected?
Hint. Look at subsets of $\mathbb{R}^2$.

(2) The aim of this exercise is to complete the proof that compactness and limit point compactness are equivalent. Let (X, d) be a limit point compact metric space. A metric space is called *separable* if it contains a countable dense subset. A collection of open sets $\{U_\alpha\}$ is called a *base* for metric space X if any open set can be covered by a sub-collection of $\{U_\alpha\}$.

(a) Show that $\mathbb{R}^k$ is separable. **Hint.** $\mathbb{Q}$ is dense in $\mathbb{R}$.

(b) Prove that X is separable.
Hint. First show for every $\delta > 0$, X can be covered by finitely many balls of radius δ. To see this, pick points $x_1, \cdots, x_n$ inductively. Having picked $x_1, \cdots x_{j-1}$, pick x_j such that $d(x_i, x_j) > \delta$ for $i = 1, \cdots j-1$, and show that this process terminates in finite steps. Then for $\delta = 1/n,\ n = 1, 2, \cdots$, look at the centers of the corresponding balls to obtain a countable dense subset.

(c) Prove that every separable metric space has a countable base.
Hint. Think of smaller and smaller balls of rational radii.

(d) Prove that every open cover of X has a *countable* sub-cover.

(e) Prove that limit point compactness implies compactness.
Hint. Suppose no finite sub-collection of the countable sub-cover $\{G_n\}$ covers X. Then show that $F_n = (G_1 \cup \cdots \cup G_n)^c$ is nonempty for all n but $\cap F_n$ is empty. Obtain a contradiction to limit point compactness.

(3) Let $l^\infty(\mathbb{C}) := \{\{a_n\}_{n=1}^\infty \mid a_n \in \mathbb{C},\ \sup_n |a_n| < \infty\}$. That is, $l^\infty(\mathbb{C})$ is the set of all bounded sequences of complex numbers. For $A = \{a_n\}, B = \{b_n\} \in X$, define

$$d(A, B) = \sup |a_n - b_n|.$$

(a) Prove that $d(A, B)$ is always finite for $A, B \in l^\infty(\mathbb{C})$, and that $(l^\infty(\mathbb{C}), d)$ is a metric space.

(b) Show that $\overline{B(0,1)} = \{\{a_n\} \mid |a_n| \leq 1,\ \text{for } n = 1, 2, \cdots\}$ is closed and bounded, but non-compact.
Hint. From the above exercise, it suffices to exhibit a sequence in $\overline{B(0,1)}$, that does not have a limit point in X. Consider $\{A_n\}$, where A_n is the sequence with one in the n^{th} place and zero everywhere else.