Assignment-10

(Due 04/19)

1. (a) Let $f_n(x) = x^n$. Show that the sequence of functions converges pointwise but not uniformly on the interval $[0, 1]$.

   (b) If $g : [0, 1] \to \mathbb{R}$ is a continuous function such that $g(1) = 0$, show that the sequence of functions $\{g(x)x^n\}_{n=1}^{\infty}$ converges uniformly on $[0, 1]$.

2. Let $C^0[0, 1]$ denote the set of all continuous real valued functions on $[0, 1]$. For $f, g \in C^0[0, 1]$, define

$$d(f, g) = \sup_{t\in[0,1]} |f(t) - g(t)|.$$

   (a) Show that d defines a metric on $C^0[0, 1]$.

   (b) Show that $f_n \to f$ in this metric, if and only if $f_n \to f$ uniformly on $[0, 1]$.

   (c) Show that $(C^0[0, 1], d)$ is a complete metric space.

3. Let sequences f_n and g_n converge uniformly on some set $E \subset \mathbb{R}$ to f and g respectively

   (a) Construct an example such that $f_n g_n$ does not converge uniformly on E.

   (b) Prove that $f_n g_n$ does converge uniformly if f and g are bounded on E.

4. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions converging uniformly to $f : [0, 1] \to \mathbb{R}$.

   (a) Suppose $\{x_m\}$ is a sequence of points in $[0, 1]$ such that $x_m \to p$, and suppose

$$A_n = \lim_{m\to\infty} f_n(x_m)$$

   exists for each n. Then show that $\{A_n\}$ converges, and that

$$\lim_{n\to\infty} A_n = \lim_{m\to\infty} f(x_m).$$

   Interpret this as a statement on interchanging limits.

   (b) Show that

$$\lim_{n\to\infty} f_n(x_n) = f(p)$$

   for all sequences $x_n \to p$. Show by providing an example that the statement is no longer true if $f_n \to f$ pointwise.

5. For $n = 1, 2, \cdots$ and $x \in \mathbb{R}$, define

$$f_n(x) = \frac{x}{1 + nx^2}.$$

   Show that $\{f_n\}$ converges uniformly to a differentiable function f on $\mathbb{R}$, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

   is correct for all $x \neq 0$ but false at $x = 0$.