MATH 104 : Mid-Term-1

22 February, 2017

Name: ______________________________

- You have 50 minutes to answer the questions.
- Use of calculators or study materials including textbooks, notes etc. is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. State whether the following statements are true or false. No explanation needed.

   (a) (2 points) Suppose $\limsup_{n\to\infty} a_n = L$, then there is a subsequence $a_{n_k} \to L$.

   (b) (2 points) $E \subset X$ is dense if and only if every non-empty open set has a non-empty intersection with E.

   (c) (2 points) Let (X, d) be a metric space, and $\{x_n\}$ a sequence in X. Then $X \setminus \{x_1, x_2, \cdots\}$, that is X with the sequence of points removed, is open.

   (d) (2 points) The set of all subsets of natural numbers is uncountable.

   (e) (2 points) In a complete metric space, closed and bounded sets are compact.

2. Find **all** the values of $p \in \mathbb{R}$ for which the following series converge. Justify your answer by quoting the relevant test.

   (a) (5 points) $\sum_{n=1}^{\infty} n^{-p}(\sqrt{n^2+1}-n)$.

   (b) (5 points) $\sum_{n=1}^{\infty} p^n n^p$.

3. An alternate distance function can be defined on $\mathbb{R}^2$ (you can take it for granted that it satisfies the axioms of a distance funciton) by the formula

$$d(\mathbf{x}, \mathbf{y}) = \max(|x_1 - y_1|, |x_2 - y_2|),$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.

(a) (4 points) Draw the ball of radius one centered at the origin in the above metric. What geometric shape do you obtain?

(b) (6 points) Are closed and bounded sets compact with this new metric? If so, give a proof. If not, give an example of a closed and bounded set (in this new metric) which is not compact.

4. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in a *complete* metric space (X, d) such that

$$d(x_{n+1}, x_n) \leq \frac{1}{2} d(x_n, x_{n-1}).$$

(a) (4 points) Show that for any $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) \leq 2^{-n} d(x_1, x_0).$$

(b) (6 points) Show that $\{x_n\}$ converges.