

6. Uniform Convergence

①

• Main problem: Let (X, d_X) and (Y, d_Y) be metric spaces.

Def: Let $E \subset X$, and $f_n: E \rightarrow Y$ be a sequence of functions. We say $\{f_n\}$ converge (pointwise) to $f: E \rightarrow Y$ if $\forall x \in E$,

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

f is then called the (pointwise) limit, and we write $f_n \rightarrow f$.

Ques) If $\{f_n\}$ are continuous and $f_n \rightarrow f$.

Then is f cont?

$$\Leftrightarrow \lim_{x \rightarrow p} f(x) = f(p)$$

$$\begin{aligned} \Leftrightarrow \lim_{x \rightarrow p} \lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} f_n(p) \\ &= \lim_{n \rightarrow \infty} \lim_{x \rightarrow p} f_n(x). \end{aligned}$$

i.e can you interchange limits?

2) Sps $f_n: [a, b] \rightarrow \mathbb{R}$, and $f_n \in R[a, b]$ $\forall n$,
 $f_n \rightarrow f$

Then is $f \in R[a, b]$? Moreover is

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$$\int_a^b \lim_{n \rightarrow \infty} f_n(t) dt = \int_a^b f(t) dt = \lim_{n \rightarrow \infty} \int_a^b f_n(t) dt ?$$

3) If $f_n: [a, b] \rightarrow \mathbb{R}$ is diff w.r.t n , $f_n \rightarrow f$. Is f diff on $[a, b]$? Is $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$?

Answer NO to all three!

Examples: 1) Let $f_n: [0, 1] \rightarrow \mathbb{R}$, $f_n(x) = x^n$

Then f_n is cont. w.r.t n . Clearly $f_n \rightarrow f$ on $[0, 1]$

where

$$f(x) = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$$

And f is not continuous.

2) $f_n(x) = n^2 x (1 - x^2)^n : [0, 1] \rightarrow \mathbb{R}$

Claim: $f_n \rightarrow 0$ on $[0, 1]$.

Pf: $f_n(0) = f_n(1) = 0$ w.r.t n . If $x \in (0, 1)$,

Then $1 - x^2 \in (0, 1)$

So $(1 - x^2)^n \xrightarrow{n \rightarrow \infty} 0$

$\Rightarrow f_n \rightarrow 0$

In this case $f \equiv 0$ and hence $f \in R[0, 1]$ but

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One can compute

$$\int_0^1 f_n(x) dx = n^2 \int_0^1 x(1-x^2)^n dx$$

$$\begin{aligned} & (1-x^2 = u) \\ & x dx = -\frac{du}{2} \quad \frac{n^2}{2} \int_0^1 u^n du \\ & = \frac{n^2}{2n+2} \xrightarrow{n \rightarrow \infty} +\infty \end{aligned}$$

But $\int_0^1 f(t) dt = \int_0^1 0 dt = 0$

So $\int_0^1 f dt \neq \lim_{n \rightarrow \infty} \int_0^1 f_n(t) dt$

3) Let $\{x_n\}_{n=1}^\infty = \mathbb{Q} \cap [0, 1]$, and consider

$$f_n(x) = \begin{cases} 1, & x \in \{x_1, \dots, x_n\} \\ 0, & \text{otherwise} \end{cases}$$

Since f_n has finitely many discontinuities $f_n \in R[0, 1] \forall n$. Moreover $f_n \rightarrow f$,

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

and $f \notin R[0, 1]$.

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4) $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ on \mathbb{R} , f_n is diff on \mathbb{R} .

Clearly $f_n \rightarrow 0$ on \mathbb{R} .

Now, $f_n'(x) = \sqrt{n} \cos(nx)$, $f_n'(0) \rightarrow \infty \neq f'(0)$

5) $f_n(x) = \sqrt{x^2 + y_n}$.

Clearly $f_n \rightarrow \sqrt{x^2} = |x| = f(x)$

Each f_n is diff on \mathbb{R} , but f is not diff at 0

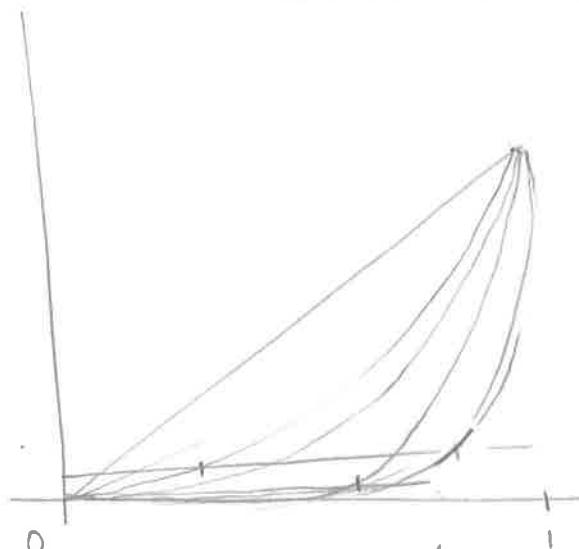
• Uniform Convergence: (X, d_X) , (Y, d_Y) metric spaces.

Sps $f_n: E \rightarrow Y$, $f_n \rightarrow f$, then $\forall x \in E \forall \epsilon > 0$,

$\exists N = N(x, \epsilon)$ s.t

$$n > N \Rightarrow d_Y(f_n(x), f(x)) < \epsilon$$

$$f_n(x) = x^n$$



As ϵ is chosen ≈ 0 , we have to choose bigger N as $x \rightarrow 1$.

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Defⁿ: We say that a seq of functions $f_n: E \rightarrow Y$ converges uniformly on E to f if $\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t.}$

$$\begin{array}{l} n > N \\ x \in E \end{array} \implies d_Y(f_n(x), f(x)) < \varepsilon.$$

We then write $f_n \xrightarrow{u.c} f$ on E .

Example: Let $f_n(x) = x^n$ on $(0, 1]$, $f_n \rightarrow 0$ on $(0, 1)$.

Claim: f_n does not converge uniformly.

Aim: For some $\varepsilon > 0$, \exists seq $\{x_n\}$ in $(0, 1)$ s.t.

$$|f_n(x_n)| > \varepsilon.$$

We will do this for some $\varepsilon > 0$, $x_n = 1 - \frac{1}{n}$.

$$\text{Then } f_n(x_n) = \left(1 - \frac{1}{n}\right)^n \rightarrow e^{-1}$$

Letting $\varepsilon < e^{-1}$ we see that $f_n(x_n) > \varepsilon$.

So $f_n \not\xrightarrow{u.c} 0$.

Rk: $f_n \not\xrightarrow{u.c} f \iff \exists \varepsilon > 0, \text{ subseq } f_{n_k} \& \text{ seq } x_k \text{ s.t.}$

$$d_Y(f_{n_k}(x_k), f(x_k)) > \varepsilon.$$

Th^m 6.1: Let $f_n: E \rightarrow Y$ be a seqⁿ of functions^⑥

s.t $f_n \rightarrow f$. Let

$$M_n = \sup_{x \in E} d_Y(f_n(x), f(x)).$$

Then $f_n \xrightarrow{u.c} f \iff \lim_{n \rightarrow \infty} M_n = 0$.

Pf: $f_n \xrightarrow{u.c} f \iff \forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t.}$

$$\forall x \in E, \quad \forall n > N, \quad d_Y(f_n(x), f(x)) < \varepsilon.$$

$\iff \forall \varepsilon > 0, \exists N \text{ s.t.}$

$$\forall n > N, \quad \sup_{x \in E} d_Y(f_n(x), f(x)) < \varepsilon.$$

$\iff \forall \varepsilon > 0, \exists N \text{ s.t.}$

$$\forall n > N, 0 < M_n < \varepsilon.$$

$\iff \lim_{n \rightarrow \infty} M_n = 0.$

Example: Again, let $f_n(x) = x^n$ on $(0, 1)$. Then $f_n \rightarrow f \equiv 0$. So here

$$M_n = \sup_{x \in (0, 1)} |f_n(x) - f(x)| = \sup_{x \in (0, 1)} x^n = 1.$$

So $M_n \not\rightarrow 0$.

$\Rightarrow f_n \not\xrightarrow{u.c} f$

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Th^m 6.2 (Cauchy criteria). Let $f_n: E \rightarrow Y$, s.p.s Y is complete. Then $\{f_n\}$ is uniformly convergent

$\Leftrightarrow \{f_n\}$ is uniformly Cauchy i.e $\forall \epsilon > 0, \exists N$ s.t

$$n, m > N \Rightarrow \sup_{x \in E} d_Y(f_n(x), f_m(x)) < \epsilon.$$

Pf: \Rightarrow S.p.s $f_n \xrightarrow{u.c} f$. Then $\forall \epsilon > 0, \exists N$ s.t

$\forall n > N, \forall x \in E$,

$$d_Y(f_n(x), f(x)) < \epsilon/2$$

Then if $n, m > N$,

$$\begin{aligned} d_Y(f_n(x), f_m(x)) &< d_Y(f_n(x), f(x)) \\ &\quad + d_Y(f_m(x), f(x)) \end{aligned}$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

\Leftarrow Clearly for each fixed $x \in E$, $\{f_n(x)\}$ is a Cauchy seqⁿ in Y .

Y complete $\Rightarrow \{f_n(x)\}$ converges. Let

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Claim: $f_n \xrightarrow{u.c} f$

Pf: let $\epsilon > 0$.

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Then $\exists N$ s.t. $n, m > N, x \in E$

$$\Rightarrow d_y(f_n(x), f_m(x)) < \varepsilon.$$

$d_y(f_n(x), y)$ is a cont. function in 'y'. So

$$d_y(f_n(x), f(x)) = \lim_{m \rightarrow \infty} d_y(f_n(x), f_m(x)) < \varepsilon.$$

$$\Rightarrow d_y(f_n(x), f(x)) < \varepsilon \quad \forall n > N, \forall x \in E$$

$$\Rightarrow f_n \xrightarrow{u.c} f$$

Uniform Convergence and Continuity

Th^m 6.3: let $f_n: E \rightarrow Y$ be cont. and sps $f_n \xrightarrow{u.s} f$.

Then $f: E \rightarrow Y$ is cont

Pf: Let $\varepsilon > 0$, and let $p \in E$. Now $f_n \xrightarrow{u.s} f \Rightarrow$

$\exists N$ s.t

$$d_y(f_N(x), f(x)) < \varepsilon/3, \forall x \in E.$$

f_N cont $\Rightarrow \exists \delta$ s.t

$$d_x(p, x) < \delta \Rightarrow d_y(f_N(x), f_N(p)) < \varepsilon/3.$$

Now if $d_x(p, x) < \delta$,

$$\begin{aligned} d_y(f(x), f(p)) &\leq d_y(f_N(x), f(x)) + d_y(f_N(x), f_N(p)) \\ &\quad + d_y(f_N(p), f(p)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \end{aligned}$$

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$$= \varepsilon$$

So

$$d_X(p, x) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon.$$

and f is cont. at p .

Thm 6.4: Let $f_n: E \rightarrow \mathbb{R}$ s.t $f_n \xrightarrow{u.c} f$. Sps
 p is a l.p. of E , and that

$$\lim_{x \rightarrow p} f_n(x) = A_n.$$

Then $\{A_n\}$ converges, $\lim_{x \rightarrow p} f(x)$ exists

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow p} f_n(x) = \lim_{n \rightarrow \infty} A_n = \lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} \liminf_{n \rightarrow \infty} f_n(x)$$

Pf: $\forall \varepsilon > 0$, $\exists N$ s.t

$$n, m > N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon.$$

$$\Rightarrow \lim_{x \rightarrow p} |f_n(x) - f_m(x)| < \varepsilon.$$

$$\Rightarrow |A_n - A_m| < \varepsilon.$$

So $\{A_n\}$ is Cauchy. \mathbb{R} complete $\Rightarrow \{A_n\}$ conv.

$$\text{Let } A = \lim_{n \rightarrow \infty} A_n.$$

Claim: $\lim_{x \rightarrow p} f(x) = A$.

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Pf: Let $\varepsilon > 0$. $\exists N_1$ s.t

$$(1) \forall x \in E, |f_N(x) - f(x)| < \varepsilon/3.$$

$$(2) |A_N - A| < \varepsilon/3.$$

Also, since $A_N = \lim_{x \rightarrow p} f_N(x)$, $\exists s > 0$ s.t

$\forall x \in E$ with $d_X(p, x) < s$, $\Rightarrow |f_N(x) - A_N| < \varepsilon/3$.

Then sps $d_X(p, x) < s$,

$$\begin{aligned} |A - f(x)| &\leq |A - A_N| + |A_N - f_N(x)| \\ &\quad + |f_N(x) - f(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

• Uniform Conv. and Integration

Th^m6.5: Let $f_n \in R[a, b]$ and $f_n \xrightarrow{u.s} f$. Then $f \in R[a, b]$.

Moreover

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} \int_a^b f_n(t) dt.$$

Pf: Let

$$M_n = \sup_{[a, b]} |f_n - f|.$$

Then $f_n \xrightarrow{u.s} f \Rightarrow \lim_{n \rightarrow \infty} M_n = 0$

By defⁿ of M_n , $\forall t \in [a, b]$,

$$|f_n(t) - f(t)| \leq M_n$$

$$\text{i.e. } f_n(t) - M_n \leq f(t) \leq f_n(t) + M_n.$$

Clearly

$$\begin{aligned} \int_a^b (f_n(t) - M_n) dt &\leq \int_a^b f(t) dt \\ (*) \quad \int_a^b f(t) dt &\leq \int_a^b [f_n(t) + M_n] dt. \end{aligned}$$

But $f_n \in R[a, b]$, and M_n constant, so

$$\begin{aligned} \int_a^b (f_n(t) - M_n) dt &= \int_a^b (f_n(t) - M_n) dt \\ \int_a^b (f_n(t) + M_n) dt &= \int_a^b (f_n(t) + M_n) dt. \end{aligned}$$

So $(*) \Rightarrow$

$$\begin{aligned} (***) \quad \int_a^b (f_n(t) - M_n) dt &\leq \int_a^b f(t) dt \leq \int_a^b f(t) dt \\ &\leq \int_a^b [f_n(t) + M_n] dt. \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \int_a^b f(t) dt - \int_a^b f(t) dt &\leq 2 \cdot \int_a^b M_n dt \\ &= 2M_n(b-a). \end{aligned}$$

Given any ϵ , $\exists M_N$ s.t $M_N < \epsilon/2(b-a)$ (12)

But then $\forall \epsilon > 0$

$$\int_a^b f(t) dt - \int_a^b f_n(t) dt < \epsilon.$$

So the two values are the same & $f \in R[a, b]$

Also $(**)$ \Rightarrow

$$\int_a^b f_n(t) dt + M_n(b-a) \leq \int_a^b f(t) dt \leq \int_a^b f_n(t) dt + M_n(b-a)$$

i.e $\left| \int_a^b f(t) dt - \int_a^b f_n(t) dt \right| \leq M_n(b-a) \rightarrow 0$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b f(t) dt$$

Uniform Convergence and Diff.

Ques: If $f_n \xrightarrow{u.s} f$ on $[a, b]$ and f_n diff on $[a, b]$, is f diff on $[a, b]$?

Ans: NO! Consider $f_n: [-1, 1] \rightarrow \mathbb{R}$

$$f_n(t) = \sqrt{t^2 + y_n}$$

Claim: $f_n \xrightarrow{u.c} |t|$ on $[-1, 1]$.

Pf.

$$\left| \sqrt{t^2 + y_n} - |t| \right| = \frac{(\sqrt{t^2 + y_n} - |t|)(\sqrt{t^2 + y_n} + |t|)}{\sqrt{t^2 + y_n} + |t|}$$

$$= \frac{y_n}{\sqrt{t^2 + y_n} + |t|}.$$

Now, $t^2, |t| \geq 0$, so $\sqrt{t^2 + y_n} + |t| > \sqrt{n}$.

So

$$\sup_{t \in [-1, 1]} \left| \sqrt{t^2 + y_n} - |t| \right| < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \rightarrow 0$$

$$\Rightarrow f_n \xrightarrow{u.c} |t|.$$

But $|t|$ is not diff.

Thm 6.6: Sps $\{f_n\}$ is a seqⁿ of functions diff on $[a, b]$. s.t

(1) $\exists t_0 \in [a, b]$ s.t $\{f_n(t_0)\}$ converges.

(2) $f_n' \xrightarrow{u.c} g$ on $[a, b]$.

Then $\exists f: [a, b] \rightarrow \mathbb{R}$ s.t $f_n \xrightarrow{u.c} f$, f diff and

$$f'(x) = g(x) \quad \forall x \in [a, b].$$

Rk: Sps. f_n' is assumed to be cont. on $[a, b]$, then a short proof can be given. By fundamental theorem,

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(t) dt$$

Sps $f_n(x_0) \rightarrow A$, and since $f_n' \xrightarrow{uc} g$,

$$\lim_{n \rightarrow \infty} f_n(x) = A + \int_{x_0}^x g(t) dt = f(x)$$

One can show $f_n \rightarrow f$ uniformly on $[a, b]$.

Pf: Let $\epsilon > 0$

Step 1: $\exists N$ s.t. $\forall n, m > N, \forall t \in [a, b]$

$$|f_n(t) - f_m(t)| < \epsilon.$$

Pf: $\exists N$ s.t.

$(f_n(t_0))$ conv) $(*) \forall n, m > N, |f_n(t_0) - f_m(t_0)| < \epsilon/2$

$(f_n' \xrightarrow{uc} g) \quad (***) |f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)} \quad \forall x \in [a, b]$

$$\text{Now, } |f_n(t) - f_m(t)| \leq |f_n(t) - f_m(t) - (f_n(t_0) - f_m(t_0))| + |f_n(t_0) - f_m(t_0)|.$$

By mean value theorem applied to $f_n - f_m$,
 $\exists c$ between t & t_0 s.t

$$\begin{aligned} & |f_n(t) - f_m(t) - (f_n(t_0) - f_m(t_0))| \\ &= |f'_n(c) - f'_m(c)| \cdot |t - t_0| \\ &\stackrel{(*)}{\leq} \frac{\epsilon}{2(b-a)} \cdot |t - t_0| < \frac{\epsilon}{2}. \end{aligned}$$

So, $n, m > N$, $t \in [a, b]$

$$|f_n(t) - f_m(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

So $\{f_n\}$ is uniformly Cauchy.
 $\Rightarrow \exists f: [a, b] \rightarrow \mathbb{R}$ s.t $f_n \xrightarrow{u.c} f$

Step 2: Fix $x \in [a, b]$, let

$$\varphi_n(t) = \frac{f_n(t) - f(x)}{t - x}, \quad \varphi(t) = \frac{f(t) - f(x)}{t - x}$$

$$f_n \text{ diff} \Rightarrow \lim_{t \rightarrow x} \varphi_n(t) = f'_n(x).$$

$$\text{Also if } t \neq x, \quad \lim_{n \rightarrow \infty} \varphi_n(t) = \varphi(t).$$

Claim: $\varphi_n \xrightarrow{u.c} \varphi$ on $[a, b] \setminus \{x\}$

Pf:

$$\begin{aligned} |\varphi_n(t) - \varphi_m(t)| &= \frac{|f_n(t) - f_m(t) - (f_n(x) - f_m(x))|}{|t - x|} \\ &= |f'_n(c) - f'_m(c)| \quad \text{for some } c \text{ between } t \text{ & } x. \\ &\stackrel{(**)}{<} \frac{\epsilon}{2(b-a)}. \end{aligned}$$

$\Rightarrow \{\varphi_n\}$ is uniformly Cauchy & hence converges uniformly.

But $\varphi_n \rightarrow \varphi$. So $\varphi_n \xrightarrow{u.c} \varphi$.

Now apply Thm 6.4, to φ_n with $E = [a, b] \setminus \{x\}$ and $A_n = f'_n(x)$, we see $\lim_{t \rightarrow x} \varphi_n(t)$ exists & hence

$$\begin{aligned} f'(x) &= \lim_{t \rightarrow x} \varphi_n(t) = \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} \varphi_n(t) \\ &= \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} \varphi_n(t) \\ &= \lim_{n \rightarrow \infty} f'_n(x) \\ &= g(x). \end{aligned}$$