

## 2.3 METRIC SPACES - SEQUENCES

Let  $(X, d)$  be a metric space.

Def<sup>n</sup>: A seq  $\{x_n\}$  in  $(X, d)$  is said to converge to  $p \in X$  if  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t.

$$n \geq N \implies d(x_n, p) < \epsilon$$

We then write

$$p = \lim_{n \rightarrow \infty} x_n$$

or simply  $x_n \rightarrow p$ . We call  $p$ , the limit of  $\{x_n\}$ . If there is no such  $p$ , we say  $\{x_n\}$  diverges.

The set of points  $\{x_n\}$  is called the range of the seq.

$\{x_n\}$  is called bounded if the range is a bounded set.

Rk 1) The range of a sequence can consist of only finitely many points

2) If  $Y \subset X$  with induced metric. A seq  $\{x_n\}$  in  $Y$  might not converge in  $Y$  but might converge in  $X$ .

Th<sup>m</sup> 2.3.1: Let  $\{x_n\}$  be a seq in  $X$ .

- (1)  $x_n \rightarrow p \iff \forall \varepsilon > 0, \exists N \text{ s.t. } x_n \in B_\varepsilon(p) \quad \forall n \geq N$ .
- (2)  $x_n \rightarrow p \& x_n \rightarrow p' \Rightarrow p = p'$ . i.e. the limit is unique.
- (3)  $\{x_n\}$  converges  $\Rightarrow \{x_n\}$  bounded.
- (4) If  $E \subset X$ . Then  $p \in X$  is a l.p of  $E \iff \exists \{x_n\}$  in  $E$  s.t.  $x_n \rightarrow p$  where  $\{x_n\}$  is an infinite set.

Pf: ① Restatement of definition!

② Sps  $p \neq p'$ . let  $\varepsilon = d(p, p') > 0$ .



$\exists N_1 \text{ s.t. } d(p, x_n) < r/4 \quad \forall n \geq N_1$

$\exists N_2 \text{ s.t. } d(p', x_n) < r/4 \quad \forall n \geq N_2$ .

If  $N = \max(N_1, N_2)$ . Then  $d(p, x_N) < r/4$   
 $d(p', x_N) < r/4$ .

$\Delta$ -wig

$$r = d(p, p') \leq d(p, x_N) + d(p', x_N) < \frac{r}{2}$$

Contradiction.

(3). Sps  $x_n \rightarrow p$ .



$\exists N$  s.t.  $\forall n > N$ ,  $d(p, x_n) < 1$ .

Let  $M = \max(1, d(p, x_1), \dots, d(p, x_{N-1}))$ .

Then  $d(x_n, p) \leq M \quad \forall n$ .

So  $\{x_n\}$  is bounded.

(4)  $\Rightarrow$   $\{p\}$  l.p.  $\Rightarrow B_{\delta}(p) \cap E \neq \emptyset \quad \forall n$ .

Let  $x_n \in B_{\delta}(p) \cap E$ . Then it is easy to see.

that  $x_n \rightarrow p$ .

$\Leftarrow$  Sps.  $\exists x_n \in E$  s.t.  $x_n \rightarrow p$ . For any  $\varepsilon > 0$ ,

$\exists N$  s.t.  $x_n \in B_\varepsilon(p) \quad \forall n \geq N$ .

In particular,  $\forall \varepsilon > 0$ ,  $B_\varepsilon(p) \cap E$  at a point other than  $p$ .

### Subsequences:

Def<sup>n</sup>: Given a sequence  $\{x_n\}$  in  $X$ , and  $\{n_k\}$  a sequence in  $\mathbb{N}$ , we call  $\{x_{n_k}\}$  a sub-sequence of  $\{x_n\}$ . We say  $x_{n_k} \rightarrow p$  if  $\forall \varepsilon > 0$ ,  $\exists K$  s.t.  $\forall k \geq K$ ,  $d(x_{n_k}, p) < \varepsilon$ .

Rk 1)  $\{x_n\}$  is a sub-sequence of itself  
 2) If  $x_n \rightarrow p \Rightarrow x_{n_k} \rightarrow p$  & subsequence  $\{x_{n_k}\}$ . Of course there might be sequences which diverge but some sub-sequence converges.  
 e.g.

$$x_n = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

If  $n_k = 2k$ ,  $x_{n_k} = 1 \neq k \Rightarrow x_{n_k} \rightarrow 1$ .

But  $\{x_n\}$  diverges.

Th<sup>m</sup> 2.3.2: 1) Let  $K \subseteq X$  be compact, and  $\{x_n\}$  a sequence in  $K$ . Then  $\exists p \in K$  s.t

$$x_{n_k} \rightarrow p$$

for some sub-sequence  $\{x_{n_k}\}$ .

2) Every bounded seq<sup>n</sup> in  $\mathbb{R}^k$  has a convergent sub-sequence

Pf: 1) CASE 1 Range of  $\{x_n\}$  finite. Then clearly there is an  $x_N$  which appears infinitely often.

Let  $n_k \in \mathbb{N}$  s.t  $x_{n_k} = x_N$ .

Then  $\{x_{n_k}\}$  trivially converges to  $x_N$ .

CASE 2: Range of  $\{x_n\}$  is infinite. Then by l.p compactness  $\exists p \in K$  which is a l.p of  $\{x_n\}$

Let  $k \in \mathbb{N}$  s.t  $x_{n_k} \in B_{y_k}(p)$

2)  $\{x_n\}$  bounded in  $\mathbb{R}^n \Rightarrow x_n \in I^k$   $\forall n$ , for some  $k$ -cell. Then apply (1)

### Cauchy Criteria & Completeness.

Def<sup>n</sup>: A seq<sup>n</sup>  $\{x_n\}$  is called Cauchy if  $\forall \epsilon > 0$ ,  $\exists N$  s.t  $\forall n, m \geq N \Rightarrow d(x_n, x_m) < \epsilon$ .

Th<sup>m</sup> 2.3.3  $\{x_n\}$  convergent  $\Rightarrow \{x_n\}$  Cauchy

Pf: Sps  $x_n \rightarrow p$

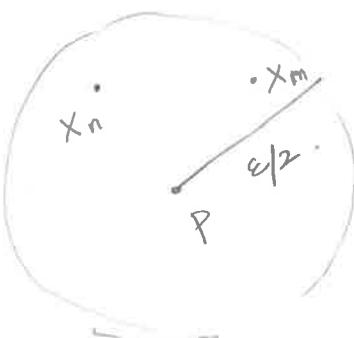
let  $\epsilon > 0$ .  $\exists N$  s.t  $\forall n \geq N$

$$d(x_n, p) < \frac{\epsilon}{2}$$

So if  $n, m \geq N$ ,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, p) + d(x_m, p) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$\Rightarrow \{x_n\}$  Cauchy.



Ques: Is the converse true?

Ans: No in general. Consider a sequence of rationals  $\{q_n\}$  s.t  $q_0 = 1$

$q_n$  = "decimal expansion of  $\sqrt{2}$  terminated upto the  $n^{\text{th}}$  place after the decimal"

i.e  $q_{n+1} = q_n + \frac{d_{n+1}}{10^{n+1}}$

where  $d_{n+1}$  is the largest integer s.t

$$q_{n+1} \leq \sqrt{2}$$

Then if  $m, n \geq N$

$$\begin{aligned}|q_m - q_n| &= \left| \frac{d_{n+1}}{10^{n+1}} + \dots + \frac{d_m}{10^m} \right| \\&\leq \frac{9}{10^{n+1}} \left( 1 + \frac{1}{10} + \dots + \frac{1}{10^{m-n-1}} \right) \\&\leq \frac{9}{10^{n+1}} \left( \frac{1}{1 - \frac{1}{10}} \right) = \frac{1}{10^n} < 10^{-N} < \epsilon\end{aligned}$$

if  $N$  is big  $\in \mathbb{Q}$ .

So  $\{q_m\}$  is Cauchy <sup>$\in \mathbb{Q}$</sup> . But  $q_n \rightarrow \sqrt{2} \notin \mathbb{Q}$ .

Def<sup>n</sup>: A metric space is called complete if every Cauchy sequence converges.

Th<sup>m</sup> 2.3.3 1) A compact metric space  $(X, d)$  is complete

2)  $\mathbb{R}^k$  is complete.

Pf: 1) Let  $\{x_n\}$  be a Cauchy sequence in  $X$ , and let  $X$  be compact. Then  $\exists p \in X$  and a sub-sequence  $x_{n_k}$  s.t.  $x_{n_k} \rightarrow p$ . (by Th<sup>m</sup> 2.3.2)

Claim:  $x_n \rightarrow p$ .

Pf: Let  $\epsilon > 0$ .

$x_{n_k} \rightarrow p \Rightarrow \exists J \text{ s.t. } \forall k > J, d(x_{n_k}, p) < \epsilon/2$

$\{x_n\}$  Cauchy  $\Rightarrow \exists N \text{ s.t. } \forall n, m \geq N$

$d(x_n, x_m) < \epsilon/2$

Since  $n_k \xrightarrow{k \rightarrow \infty} \infty$ ,  $\exists k > J$  s.t.  $n_k > N$

Then  $d(x_n, x_{n_k}) < \epsilon/2$

$\Delta$ -ineq  $\Rightarrow$

$$d(x_n, p) \leq d(x_n, x_{n_k}) + d(x_{n_k}, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So  $\exists N$  s.t.  $\forall n > N$

$$d(x_n, p) < \epsilon$$

and claim is proved.

(2) Let  $\{\vec{x}_n\}$  Cauchy seq<sup>n</sup> in  $\mathbb{R}^k$

Claim:  $\{\vec{x}_n\}$  - bounded.

Pf.  $\exists N$  s.t.  $\forall n > N$ ,

$$d(\vec{x}_n, \vec{x}_N) < 1.$$



Let  $R = \max(1, d(\vec{x}_N, \vec{x}_1), \dots, d(\vec{x}_N, \vec{x}_{N-1}))$

Then  $\{\vec{x}_n\} \subset B_R(\vec{x}_N)$ .

So  $\{\vec{x}_n\}$  bounded

Then Thm 2.3.2  $\Rightarrow \exists$  sub-seq  $\{\vec{x}_{n_k}\}$

and  $\vec{p} \in \mathbb{R}^k$  s.t.  $\vec{x}_{n_k} \rightarrow \vec{p}$

By the same argument as above.  
 $\vec{x}_n \rightarrow p$

In the course of the above proof we actually proved the foll. useful facts about Cauchy sequences.

Th<sup>m</sup> 2.3.4: Let  $\{x_n\}$  be a Cauchy seq<sup>n</sup> in any metric sp.  $(X, d)$ . Then

(1)  $\{x_n\}$  is bounded.

(2) If  $x_{n_k} \rightarrow p$  for some sub-sequence we have  $x_n \rightarrow p$ .

Rk: To prove convergence in  $\mathbb{R}^k$ , we only need to check that the sequence is Cauchy.

Sequences in  $\mathbb{R}^k$

Th<sup>m</sup> 2.3.5: If  $\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,k})$  is a sequence in  $\mathbb{R}^k$  Then  $\vec{x}_n \rightarrow \vec{a} = (a_1, \dots, a_k)$   
 $\iff x_{n,j} \rightarrow a_j \quad \forall j = 1, 2, \dots, k$ .

Lemma 2.3.6: If  $\vec{u} = (u_1, \dots, u_k)$ ,  $\vec{v} = (v_1, \dots, v_k)$   
 $\in \mathbb{R}^k$ ,

$$\max_{j=1, \dots, k} |u_j - v_j| \leq |\vec{u} - \vec{v}| \leq \sqrt{k} \cdot \max_{j=1, \dots, k} |u_j - v_j|$$

where recall that

$$|\vec{u} - \vec{v}| = \left( \sum_{j=1}^k (u_j - v_j)^2 \right)^{\frac{1}{2}}$$

Pf: Let  $M = \max_{j=1, \dots, k} |u_j - v_j|$

Since  $|u_j - v_j| < M \quad \forall j$ ,

$$|\vec{u} - \vec{v}|^2 = \sum_{j=1}^k (u_j - v_j)^2 \leq k \cdot M$$

$$\Rightarrow |\vec{u} - \vec{v}| < \sqrt{k} \cdot M$$

On the other hand,

$$|\vec{u} - \vec{v}|^2 \geq (u_j - v_j)^2 \quad \forall j$$

$$\Rightarrow |\vec{u} - \vec{v}|^2 \geq M^2$$

Pf of Th<sup>m</sup>.

$\Rightarrow$  Sps  $\vec{x}_n - \vec{a}$ . Then  $\forall \epsilon > 0$ ,  $\exists N$  s.t  
 $n > N \Rightarrow |\vec{x}_n - \vec{a}| < \epsilon$ .

$\Rightarrow$  for each  $j$ ,  $|x_{n,j} - a_j| < \epsilon$ ,  $\forall n > N$

So for each  $j$ ,  $x_{n,j} \xrightarrow{n \rightarrow \infty} a_j$

$\Leftarrow$  Sps  $x_{n,j} \rightarrow a_j \quad \forall j$ . For  $\epsilon > 0$ ,  $\exists N_j$  s.t  
 $n > N_j \Rightarrow |x_{n,j} - a_j| < \frac{\epsilon}{\sqrt{k}}$

Let  $N = \max(N_1, N_2, \dots, N_k)$ .

$$n > N \Rightarrow |x_{n,j} - a_j| < \frac{\epsilon}{\sqrt{k}} \quad \forall j = 1, 2, \dots, k.$$

By Lemma,

$$\begin{aligned} |\vec{x}_n - \vec{a}| &< \sqrt{k} \cdot \max_{j=1, \dots, k} |x_{n,j} - a_j| \\ &< \frac{\epsilon}{\sqrt{k}} \cdot \sqrt{k} = \epsilon. \end{aligned}$$

So to study convergence in  $\mathbb{R}^k$ , it is enough to study convergence in  $\mathbb{R}$ .

Thm 2.3.7: Let  $\{s_n\}$  and  $\{t_n\}$  sequences in  $\mathbb{R}$ . s.t  $s_n \rightarrow s$ ,  $t_n \rightarrow t$ . Then

$$(1) \lim_{n \rightarrow \infty} s_n \pm t_n = s \pm t.$$

$$(2) \lim_{n \rightarrow \infty} c \cdot s_n = c \cdot s, \quad \lim_{n \rightarrow \infty} c \pm s_n = c + s \quad \forall c \in \mathbb{R}$$

$$(3) \lim_{n \rightarrow \infty} s_n \cdot t_n = s \cdot t.$$

$$(4) \lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}, \quad s_n \neq 0 \quad \forall n \text{ & } s \neq 0.$$

$$(5) \text{ (Squeeze principle): If } s_n \leq r_n \leq t_n, \quad s = t, \\ \Rightarrow r_n \rightarrow s.$$

$$(6) \text{ If } s_n \geq A \quad \forall n \geq N, \text{ Then } s \geq A \\ s_n \leq B \quad \forall n \geq N, \text{ Then } s \leq B.$$

Pf: (1), (2), (3), (6) in book.

(3). We use the identity

$$\begin{aligned} s_n t_n - st &= s_n t_n - s_n t + s_n t - st \\ &= s_n(t_n - t) + (s_n - s)t \end{aligned}$$

Since  $s_n \rightarrow s$ ,  $t_n \rightarrow t$ , we know that  $\{s_n\}$  and  $\{t_n\}$  are bounded. So  $\exists M$  s.t

$$|s_n|, |t_n| < M \quad \forall n,$$

$$|s|, |t| < M.$$

Now

$$\begin{aligned} |s_n t_n - st| &\leq |s_n(t_n - t)| + |(s_n - s)t| \\ &\leq M|t_n - t| + |s_n - s|M \end{aligned}$$

Given  $\epsilon > 0$ ,  $\exists N$  s.t.

$$n > N \Rightarrow |t_n - t| < \frac{\epsilon}{2M}$$

$$|s_n - s| < \frac{\epsilon}{2M}$$

$$\begin{aligned} \text{So } n > N &\Rightarrow |s_n t_n - st| < M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &= \epsilon. \end{aligned}$$

(4). Since  $s_n \rightarrow s \neq 0$ , applying def<sup>n</sup> of conv. with  $\epsilon = |s|/2 > 0$ ,  $\exists N_1$  s.t

$$n > N_1 \Rightarrow |s_n - s| < \frac{|s|}{2}.$$

$$\text{Then } \boxed{|s_n| \geq \frac{|s|}{2} \quad \forall n \geq N_1}$$

else by  $\Delta$ -ineq  $|s| \leq |s_n - s| + |s_n|$   
 $< \frac{|s|}{2} + \frac{|s|}{2} = |s|$ .  
contradiction!

Now, if  $n > N_1$ .

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s_n - s|}{|s_n \cdot s|} \leq \frac{2|s_n - s|}{|s|^2}$$

$\exists N_2$  s.t.  $\forall n > N_2$ ,  $|s_n - s| < \varepsilon \cdot \frac{|s|^2}{2}$ .

if  $N = \max(N_1, N_2)$ , then

$$n \geq N \Rightarrow \left| \frac{1}{s_n} - \frac{1}{s} \right| < \varepsilon$$

$$(5) \quad s_n, t_n \rightarrow s \Rightarrow \text{Given } \varepsilon > 0, \exists N \text{ s.t.}$$

$$n > N \Rightarrow |s_n - s| < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon$$

$$|t_n - s| < \varepsilon \iff s - \varepsilon < t_n < s + \varepsilon$$

Since  $t_n < r_n < s_n$ ,

$$s - \varepsilon < r_n < s + \varepsilon \quad \forall n > N,$$

$$\Rightarrow |r_n - s| < \varepsilon \quad \forall n > N.$$

So  $r_n \rightarrow s$ .

## • Monotonic Sequences

Def<sup>n</sup>: Let  $\{a_n\}$  be a sequence in  $\mathbb{R}$ .

- 1) It is called increasing, written as  $a_n \nearrow$   
if  $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$ .
- 2) It is called decreasing, written as  $a_n \searrow$   
if  $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$ .
- 3) It is called monotonic if increasing  
or decreasing.

Def<sup>n</sup> (Convergence to infinity): Let  $\{a_n\}$  seq<sup>n</sup> in  $\mathbb{R}$ . We say

- 1)  $a_n \rightarrow \infty$  if  $\forall M, \exists N$  s.t.  
 $a_n \geq M \quad \forall n \geq N$ .
- 2)  $a_n \rightarrow -\infty$  if  $\forall M, \exists N$  s.t.  $\forall n \geq N$ ,  
 $a_n \leq -M$ .

Rk: If we simply say  $a_n$  converges, we usually mean to a finite #.

Def<sup>n</sup>: Let  $E$  be a subset of  $\mathbb{R}$ .

1) If  $E$  has no upper bound we define

$$\sup E = \infty.$$

2) If  $E$  has no lower bound, we define

$$\inf E = -\infty.$$

Th<sup>m</sup> 2.3.8: 1) If  $\{a_n\}$  is an increasing seq<sup>n</sup>,

then  $a_n \rightarrow \sup_n \{a_n\}$

2) If  $\{a_n\}$  is a decreasing seq<sup>n</sup>; then  
 $a_n \rightarrow \inf_n \{a_n\}$

Pf: 1) CASE 1:  $\sup a_n = \infty$

Given  $M > 0$ ,  $\exists N$  s.t.  $a_N \geq M$

Since  $\{a_n\}$  increasing,  $\forall n \geq N$ ,

$$a_n \geq a_N \geq M.$$

So  $a_n \rightarrow \infty$ .

CASE 2:  $\sup a_n = \alpha < \infty$



Let  $\varepsilon > 0$ . Since  $\alpha - \varepsilon$  cannot be an u.b for  $\{a_n\}$  (since  $\alpha$  is the l.u.b).

$\exists N \text{ s.t}$

$$a_N \geq \alpha - \varepsilon.$$

$\{a_n\}$  increasing  $\Rightarrow a_n \geq \alpha - \varepsilon \quad \forall n \geq N$ .

On the other hand,  $a_n \leq \alpha + \varepsilon \quad \forall n$ .

So for  $n \geq N$ ,  $\alpha - \varepsilon \leq a_n \leq \alpha$

In particular  $n \geq N \Rightarrow |a_n - \alpha| < \varepsilon$ .

2) Similar proof.

Example: Consider the seq.,  $a_0 = \sqrt{2}$

$$a_{n+1} = \sqrt{2 + a_n}$$

Claim 1  $a_n \leq 2 \quad \forall n$ .

Pf: Induction. True for  $n=0$ .

Sps proved for  $n$ , Then

$$a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2+2} = 2.$$

Claim 2  $a_n \uparrow$

$$\begin{aligned} \text{Pf. } a_{n+1} &= \sqrt{2+a_n} \stackrel{(2 \geq a_n)}{\geq} \sqrt{a_n+a_n} = \sqrt{2a_n} \\ &\stackrel{(2 \geq a_n)}{\geq} \sqrt{a_n \cdot a_n} = a_n. \end{aligned}$$

So  $a_n \uparrow$  and bounded above. Hence  $a_n \rightarrow \alpha < \infty$ . Once we learn about continuity, we will see that  $\alpha$  satisfies

$$\alpha = \sqrt{2+\alpha}$$

$$\Rightarrow \alpha^2 = \alpha - 2 = 0$$

$$\Rightarrow \alpha = 1 \text{ or } 2.$$

Since  $a_0 = \sqrt{2} > 1$  and  $a_n \uparrow$ ,

$$\alpha \neq 1. \text{ So } \alpha = 2$$

### • lim sup & lim inf

Let  $\{a_n\}$  be a sequence in  $\mathbb{R}$ , and

let

$L = \{p \in \mathbb{R} \cup \{\pm\infty\} \mid a_{n_k} \rightarrow p \text{ for some sub-sequence } a_{n_k}\}$ .

i.e  $L$  is the set of all sub-sequential limits (including  $\pm\infty$ ).

Note that  $L$  will be bounded if  $\{a_n\}$  is a bounded seq<sup>n</sup>.

Def<sup>n</sup> 1) The limit superior of  $\{a_n\}$  is defined as

$$\limsup_{n \rightarrow \infty} a_n = \sup L$$

2) The limit inferior is defined as.

$$\liminf_{n \rightarrow \infty} a_n = \inf L$$

Rk:  $\limsup_{n \rightarrow \infty} a_n = \infty \Leftrightarrow \{a_n\}$  not bounded above.

$\liminf_{n \rightarrow \infty} a_n = -\infty \Leftrightarrow \{a_n\}$  not bounded below.

Example:  $a_n = (-1)^n$ . Then

$$L = \{-1, 1\}$$

So  $\limsup_{n \rightarrow \infty} a_n = 1$ ,  $\liminf_{n \rightarrow \infty} a_n = -1$ .

Th<sup>m</sup> 2.3.9: Let  $\{a_n\}$  be a bounded seq<sup>n</sup>.

1)  $a^* = \limsup a_n \Leftrightarrow$  the foll holds.  
(a)  $\forall \varepsilon > 0$ ,  $\exists N$  s.t.  $\forall n \geq N$ ,

$$a_n \leq a^* + \varepsilon$$

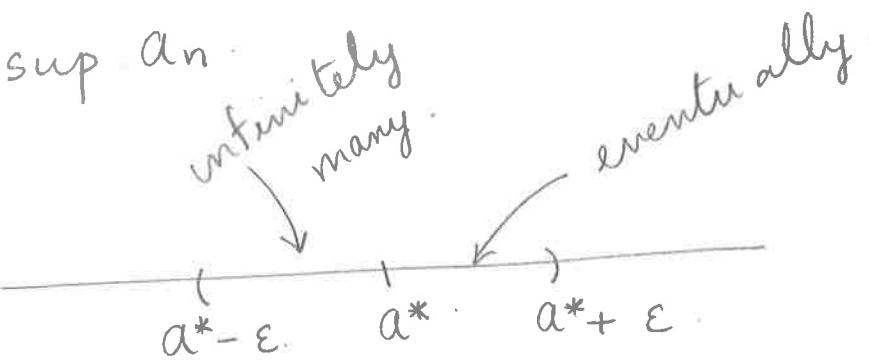
(b)  $\forall \varepsilon > 0$ ,  $\forall N$ ,  $\exists n \geq N$  s.t.

$$a_n \geq a^* - \varepsilon.$$

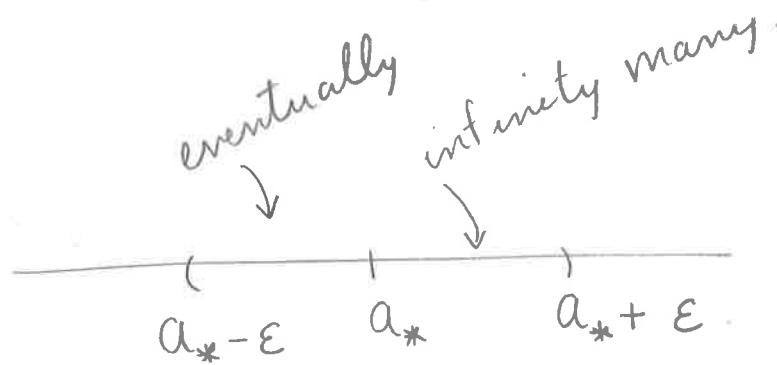
2).  $a_* = \liminf a_n \iff$  (a)  $\forall \varepsilon > 0$ ,  $\exists N$  s.t.  
 $n \geq N \Rightarrow a_n \geq a_* - \varepsilon.$

(b)  $\forall \varepsilon > 0$ ,  $\forall N$ ,  $\exists n \geq N$  s.t.  
 $a_n \leq a_* + \varepsilon.$

Rk: 1)  $a^* = \limsup a_n$



2)  $a_* = \liminf a_n$



Rk: Clearly  $a_* \leq a^*$

Cor 2.3.10:  $\{a_n\}$  converges  $\iff$

$$\liminf a_n = \limsup a_n.$$

In such a case:

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Pf.  $\Rightarrow$  Sps  $a_n \rightarrow a$ . Then  $a_{n_k} \rightarrow a$   
+ sub-sequences  $a_{n_k}$ . So  $L = \{a\}$ .  
 $\Rightarrow \sup L = \inf L = a$ . Done!

$\Leftarrow$  Sps  $\liminf a_n = \limsup a_n = a$ .

Claim:  $a_n \rightarrow a$ .

Pf: Let  $\epsilon > 0$ . By Thm 1(a).  $\exists N_1$  s.t  
 $a_n \leq a + \epsilon \quad \forall n \geq N_1$ .

2(a)  $\exists N_2$  s.t  $a_n \geq a - \epsilon \quad \forall n \geq N_2$ .

if  $N = \max(N_1, N_2)$ . Then

$$a - \epsilon \leq a_n \leq a + \epsilon \quad \forall n \geq N$$

$$\Rightarrow |a_n - a| < \epsilon \quad \forall n \geq N$$

So  $a_n \rightarrow a$ .

Pf of Th<sup>n</sup>: i)  $\Rightarrow$  let  $a^* = \limsup a_n$

and  $L$  as before i.e.  $a^* = \sup L$ .

Let  $\epsilon > 0$ . Sps, (a) does not hold.

Then  $\nexists k$ ;  $\exists n_k > k$  s.t.

$$a_{n_k} > a^* + \epsilon.$$

Then since  $\{a_n\}$  bounded  $\Rightarrow \{a_{n_k}\}$  bounded.

So  $\exists$  sub-sub-sequence  $a_{n_{k_e}} \rightarrow p \in R$

But then  $p > a^* + \epsilon$ ,  $p \in L$ .

Contradiction since  $a^*$  u.b for  $L$ .

Sps (b) does not hold. Then  $\exists N$   
s.t.  $\forall n \geq N$ ,

$$a_n < a^* - \epsilon.$$

Then for any sub-sequence  $a_{n_k}$ ,  $\exists J$

$a_{n_k} < a^* - \epsilon \quad \forall k > J$ .

So if  $a_{n_k} \rightarrow p$ , then  $p < a^* - \epsilon$ .

i.e.  $a^* - \epsilon$  is an u.b for  $L$ .

Contradiction, since  $a^*$  is l.u.b.

$\Leftarrow$  (b) Sps  $a^* < \limsup a_n$ .

$$\text{---} + ) + \text{---} \\ a^* \quad a^* + \varepsilon. \limsup a_n$$

Let  $\varepsilon > 0$ , s.t.  $a^* + \varepsilon < \limsup a_n$ .

If  $p \in L$ , then  $a_{n_k} \rightarrow p$  for some  $\{a_{n_k}\}$ .

(a)  $\Rightarrow \exists J$  s.t.  $a_{n_k} < a^* + \varepsilon$

$$\Rightarrow p < a^* + \varepsilon.$$

So  $a^* + \varepsilon$  an ub for  $L$ .

Contradiction since  $\limsup a_n = \sup L > a^* + \varepsilon$ .

Sps  $a^* > \limsup a_n$ .

$$\text{---} + \text{---} + \text{---} \\ \limsup a_n \quad a^* - \varepsilon \quad a^*$$

*infinitely many*

Then  $\exists$  sub-seq  $a_{n_k}$  s.t.  $a_{n_k} \rightarrow p$ .

and  $p > a^* - \varepsilon$ . So  $p \in L$  but  $p > \sup L$   
 $> \sup L$  Contradiction.

We end with a theorem on some special sequences. The proof relies on the binomial theorem. See Rudin (Th<sup>m</sup> 3.20) for details.

Th<sup>m</sup> 2.3.11: (a)  $p > 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ .

(b)  $p > 0 \Rightarrow \lim_{n \rightarrow \infty} p^{y_n} = 1$ .

(c)  $\lim_{n \rightarrow \infty} n^{y_n} = 1$ .

(d).  $p > 0$  and  $\alpha \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$ .

(i.e exponential trumps any power of  $n$ ).

(e)  $|x| < 1 \Rightarrow \lim_{n \rightarrow \infty} x^n = 0$

