

• Series of functions: let (X, d) metric space.

Defⁿ: Given a seqⁿ of functions $f_n: E \rightarrow \mathbb{R}$, $E \subset X$, we define the seqⁿ of partial sums by $S_n: E \rightarrow \mathbb{R}$,

$$S_n(x) = \sum_{k=0}^n f_k(x).$$

We say that $\sum f_n$ converges pointwise or uniformly if $\{S_n\}$ converges pointwise or uniformly resp. In either case, we write

$$\sum_{n=0}^{\infty} f_n(x) = \lim_{n \rightarrow \infty} S_n(x).$$

Th^m: (Cauchy criteria). $\sum f_n$ converges uniformly if and only if $\forall \epsilon, \exists N$ s.t. $\forall n, m > N, \forall x \in E$

$$\left| \sum_{k=n}^m f_k(x) \right| < \epsilon.$$

Pf: Follows from uniform Cauchy criteria applied to $\{S_n\}$.

Th^m (Weierstrass M-test): Sp's $\{f_n\}$ is a seqⁿ of functions, $f_n: E \rightarrow \mathbb{R}$, and $\{M_n\}$ seq of positive reals s.t. $|f_n(x)| \leq M_n \quad \forall x \in E, \forall n$.

Then

$\sum M_n$ converges $\Rightarrow \sum f_n$ converges uniformly

Pf: $\sum M_n$ conv. $\Rightarrow \forall \varepsilon, \exists N$ st $\forall n, m > N$

$$\sum_{k=n}^m M_k < \varepsilon.$$

But $\forall x \in E$,

$$\left| \sum_{k=n}^m f_k(x) \right| \leq \sum_{k=n}^m |f_k(x)| \leq \sum_{k=n}^m M_k < \varepsilon.$$

So by Th^m, $\sum f_n$ converges uniformly.

Example (Fourier series). Consider $\sum a_n \sin(nx)$.
Sp. $\sum |a_n|$ converges. Then since $|\sin(nx)| \leq 1$,
 $|a_n \sin(nx)| \leq |a_n|$. So by theorem,

$\sum |a_n|$ converges $\Rightarrow \sum a_n \sin(nx)$ converges uniformly.

Th^m: ① If $f_n: E \rightarrow \mathbb{R}$ is cont., and $\sum f_n$ converges uniformly, then $\sum f_n$ is cont.

② If $f_n \in R[a, b] \forall n$, and $\sum f_n$ conv. uniformly,

$\Rightarrow f = \sum f_n \in R[a, b]$, and

$$\int_a^b f(t) dt = \sum_{n=0}^{\infty} \int_a^b f_n(t) dt$$

③ If $f_n: [a, b] \rightarrow \mathbb{R}$ diff s.t $\sum f_n(t_0)$ converges uniformly for some $t_0 \in [a, b]$, and $\{f_n'\}$ converges uniformly on $[a, b]$. Then $\{f_n\}$ conv. uniformly on $[a, b]$ and furthermore

$$\frac{d}{dt} \sum_{n=0}^{\infty} f_n(t) = \sum_{n=0}^{\infty} f_n'(t).$$

Pf (3). Consider the sequence $\{S_n\}$. Then

$$S_n'(t) = \sum_{k=0}^n f_k'(t).$$

Since $\{S_n(t_0)\}$ converges, and $\{S_n'\}$ converges uniformly, by Thm on uniform convergence and differentiation $\{S_n\}$ converges uniformly on $[a, b]$. Moreover if $s = \lim_{n \rightarrow \infty} S_n = \sum f_n$,

$$\begin{aligned} s'(t) &= \lim_{n \rightarrow \infty} S_n'(t) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n f_k'(t) \\ &= \sum_{k=0}^{\infty} f_k'(t). \end{aligned}$$

Power Series

Defn: Given a seqⁿ $\{C_n\}$ in \mathbb{R} , the series

$$\sum_{n=0}^{\infty} C_n(x-a)^n$$

is called a power series centered at 'a'. The numbers c_n are called coefficients of the power series.

Th^m (Fundamental theorem of power series)

- 1) The series converges absolutely on $|x-a| < R$ and diverges on $|x-a| > R$, where

$$R = \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}$$

R is called the radius of convergence.

- 2) Convergence is uniform on $|x-a| \leq r \neq R$

- 3) For $|x-a| < R$ if we write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Then f is diff on $(a-R, a+R)$. Moreover

$$f'(x) = \sum_{n=1}^{\infty} n \cdot c_n (x-a)^{n-1}$$

Rk: The above theorem does not tell us anything about convergence at $x = a-R$ or $a+R$.

In any case, the set

$$I = \{x \in \mathbb{R} \mid \sum c_n(x-a)^n \text{ conv.}\}$$

is an interval, called the interval of convergence.
It is either $(a-R, a+R)$, $[a-R, a+R]$, $(a-R, a+R]$ or $[a-R, a+R]$.

Pf of Fundamental Th^m. Without loss of generality, let $\boxed{a=0}$.

i) Let $a_n = c_n x^n$. Now

$$\begin{aligned}\limsup_{n \rightarrow \infty} |a_n|^{1/n} &= \limsup_{n \rightarrow \infty} |c_n x^n|^{1/n} \\ &= |x| \limsup_{n \rightarrow \infty} |c_n|^{1/n} \\ &= \frac{|x|}{R}.\end{aligned}$$

By root test

$\sum a_n$ converges absolutely if $\frac{|x|}{R} < 1$
or $|x| < R$.

diverges if $\frac{|x|}{R} > 1$ or $|x| > R$.

② Let $r < R$. Then on $|x| \leq r$.

$$|c_n x^n| \leq C_n r^n.$$

Root test $\Rightarrow \sum c_n r^n$ converges (since $r < R$).

M-test $\Rightarrow \sum c_n x^n$ conv. uniformly on $|x| \leq r$.

③ Apply theorem on uniform series conv. & differentiation. Let $|x| \leq r < R$, and

$$(1) \quad S_N(x) = \sum_{n=0}^N c_n x^n \quad (N \in \mathbb{N}, N \geq 0).$$

Then S_N is a polynomial of deg ' N ' and

$$S'_N(x) = \sum_{n=1}^N n \cdot c_n x^{n-1}$$

Claim: (a). $\{S_n(x)\}$ converges for all $x \in (-r, r)$

(b) $\{S'_N(x)\}$ converges uniformly on $|x| \leq r$.

Sps Claim is proved, then by the theorem on uniform conv. & differentiation, if

$$f(x) = \lim_{N \rightarrow \infty} S_N(x) = \sum_{n=0}^{\infty} c_n x^n, \text{ then}$$

f is differentiable on $|x| \leq r$, and.

$$f'(x) = \lim_{N \rightarrow \infty} S_N'(x).$$

$$= \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

Since f is diff on $|x| \leq r$ & $r < R$

$\Rightarrow f$ is diff on $|x| < R$ &

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

proving (3) in Th^m.

Pf of Claim (a) follows from (1) in Th^m.

(b) On $|x| \leq r$.

$$|n c_n x^{n-1}| \leq n |c_n| \cdot r^{n-1}.$$

Consider $\sum_{n=1}^{\infty} n |c_n| r^{n-1}$. Then

$$\limsup_{n \rightarrow \infty} (n |c_n| r^{n-1})^{y_n} = \limsup_{n \rightarrow \infty} n^{y_n} |c_n|^{y_n} r^{n-1-y_n}$$

$$= \lim_{n \rightarrow \infty} n^{y_n} r^{1-y_n} \cdot \limsup_{n \rightarrow \infty} |c_n|^{y_n}$$

$$= \frac{r}{R} < 1.$$

So $\sum_{n=1}^{\infty} n C_n z^{n-1}$ converges.

M-test $\Rightarrow \sum_{n=1}^{\infty} n C_n x^{n-1}$ converges uniformly
on $|x| \leq r$.

Claim is proved!

Cor: The radius of convergence R of $\sum C_n(x-a)$
satisfies

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{C_{n+1}}{C_n} \right|$$

if the limit exists

Pf: By assignment

$$\liminf_{n \rightarrow \infty} \left| \frac{C_{n+1}}{C_n} \right| \leq \limsup_{n \rightarrow \infty} |C_n|^{1/n} \leq \limsup_{n \rightarrow \infty} \left| \frac{C_{n+1}}{C_n} \right|$$

So if limit exist, then

$$\lim_{n \rightarrow \infty} \left| \frac{C_{n+1}}{C_n} \right| = \limsup_{n \rightarrow \infty} |C_n|^{1/n} = \frac{1}{R}$$

Examples: ① $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Since $\lim_{n \rightarrow \infty} \frac{1}{n^n} = 1$.

$R=1$. So series converges on $|x| < 1$.

Boundary points: $x=1$, $\sum \frac{1}{n}$ diverges

$x=-1$ $\sum \frac{(-1)^n}{n}$ converges.

So, interval of convergence $I = [-1, 1]$.

② Exponential: $\sum_{n=1}^{\infty} \frac{x^n}{n!}$. Easier to use

ratio test (Cor). $c_n = y n!$

$$\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

So $R = \infty$.

So $I = (-\infty, \infty)$.

Defⁿ: We define

$$\boxed{\exp(x) = e^x = \sum \frac{x^n}{n!}}$$

Note $e^0 = 1$.

By Theorem part (3), e^x is diff on \mathbb{R} ,
and

$$\begin{aligned}\frac{d}{dx} e^x &= \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}\end{aligned}$$

$$\stackrel{n-1=m}{=} \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x$$

So $(e^x)' = e^x$.

Fact: In fact one can show that $A \cdot e^x$ is the unique function satisfying

$$\left\{ \begin{array}{l} f'(x) = f(x) \\ f(0) = A \end{array} \right.$$

③ Sine / Cosine:

Defⁿ: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \dots$$

Check: Both power series have $R = \infty$

$$\text{Thm} \Rightarrow \frac{d}{dx} \sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(2n+1) \cdot x^{2n}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = \cos(x)$$
$$\frac{d}{dx} \cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot (2n) \frac{x^{2n-1}}{(2n)!}$$
$$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n-1}}{(2n-1)!}$$
$$\stackrel{n-1=m}{=} \sum_{m=0}^{\infty} (-1)^{m+1} \cdot \frac{x^{2m+1}}{(2m+1)!}$$
$$= -\sin(x)$$

Fact: $\sin(x)$ & $\cos(x)$ are periodic with the same period, which we denote by τ . i.e

$$\sin(x + \tau) = \sin(x)$$

$$\cos(x + \tau) = \cos(x)$$

We define π to be $\pi = \tau/2$

④ Geometric series: $\sum_{n=0}^{\infty} x^n$, $c_n = 1$, $a = 0$

$$R = \limsup_{n \rightarrow \infty} |x^n|^{1/n} = 1.$$

Clearly divergence at $x = -1, 1$. In fact we have already seen that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

By differentiating, we can see:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots$$

Cor: Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on $|x-a| < R$. Then

$$\int_a^x f(t) dt = \sum_{n=0}^{\infty} c_n \frac{(x-a)^n}{(n+1)}$$

on $|x-a| < R$.

Pf: Let $S_N(t) = \sum_{n=0}^N c_n(t-a)^n$ on $|t-a| < R$.

Since $S_N \xrightarrow{u.c} f$, for any $x \in (a-R, a+R)$

$$\int_a^x f(t) dt = \lim_{N \rightarrow \infty} \int_a^x S_N(t) dt$$

But

$$\int_a^x S_N(t) dt = \sum_{n=0}^N C_n \frac{(t-a)^{n+1}}{n+1} \Big|_a^x \\ = \sum_{n=0}^N C_n \frac{(x-a)^{n+1}}{n+1}$$

So

$$\int_a^x f(t) dt = \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1}$$

Rk: In terms of indefinite integrals

$$\int f(x) dx = A + \sum_{n=0}^{\infty} \frac{C_n (x-a)^{n+1}}{(n+1)}$$

Example: 1) $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^n, |x| < 1$

Integrating

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

To prove

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots}$$

$$2) \quad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

$$\tan^{-1}(x) = A + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

$$\tan^{-1}(0) = 0, \quad \text{So } A = 0$$

$$\boxed{\tan^{-1}(x) = x - \frac{x^3}{3} + \dots}$$

Taylor Series of a function

Ques: When is a function represented by a power series?

Prop: Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ on $|x-a| < R$.

Then $f \in C^\infty(a-R, a+R)$ i.e. f is infinitely diff on $(a-R, a+R)$. Moreover

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n \cdot (n-1) \dots (n-m+1) c_n (x-a)^{n-m}$$

In particular $\boxed{c_m = f^{(m)}(a)/m!}$

Pf: We proceed by induction.

Base Case: $m=0, 1$.

When $m=0$, $f(a) = C_0$:

$m=1$, Thm $\Rightarrow f'$ exists and

$$f'(x) = \sum_{n=1}^{\infty} n \cdot C_n (x-a)^{n-1}$$

$$= C_1 + 2C_2(x-a) + \mathcal{O}(|x-a|^2).$$

So $f'(a) = C_1$

Inductive Step Sps Prop proved for $m=1, \dots, N$.

i.e

$$f^{(N)}(x) = \sum_{n=N}^{\infty} n(n-1)\dots(n-N+1) C_n (x-a)^{n-N}$$

Again by Thm, since $f^{(N)}(x)$ is a power series
on $(a-R, a+R)$ it is diff, and

$$f^{(N+1)}(x) = \frac{d}{dx} f^{(N)}(x) = \sum_{n=N}^{\infty} n(n-1)\dots(n-N+1)(n-N) C_n \cdot x^{n-N-1}.$$

$$= \sum_{n=N+1}^{\infty} n(n-1)\dots(n-N) C_n x^{n-N-1}.$$

~~REPEAT~~

Plugging in $x = a$,

$$f^{(N+1)}(a) = (N+1) \cdots \cdot 1 \cdot C_{N+1}.$$

So $C_{N+1} = \frac{f^{(N+1)}(a)}{(N+1)!}$

Ques: Given an $f \in C^\infty(a-R, a+R)$ does f have a power series expansion.

Defⁿ: For an $f \in C^\infty(a-R, a+R)$, the Taylor series at $x = a$ is defined by

$$T_f(x; a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Ques: Given $f \in C^\infty(a-R, a+R)$, does $T_f(x)$ converge on $(a-R, a+R)$. If so, is $f(x) = T_f(x; a)$?

Ans: ~~No!~~ No! to both.

i) $f(x) = \frac{1}{1+x^2}$ on \mathbb{R} , $f \in C^\infty(-\infty, \infty)$.

But $T_f(x; 0) = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges only

on $|x| < 1$.

$$\Rightarrow f(x) = \begin{cases} e^{-y x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Check: $f \in C^\infty(-\infty, \infty)$ and $f^{(n)}(0) = 0 \neq n$.

So $T_f(x; 0) = 0$, but $f(x) \neq 0 \neq x \neq 0$.

So $f(x) \neq T_f(x; 0)$.

• Recall the N^{th} Taylor polynomial is defined as:

$$T_N(x; a) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n, |x-a| < R$$

and the remainder,

$$R_N(x; a) := f(x) - T_N(x; a).$$

$$= \frac{f^{(n+1)}(c_x)}{(n+1)!} (x-a)^{n+1}$$

for some $c_x \in (a-R, a+R)$.

Clearly $f(x) = T_f(x; a) \iff \lim_{N \rightarrow \infty} R_N(x; a) = 0$.

Th^m: Sps $\exists M$ s.t. $\forall x \in (a-R, a+R)$.

$$|f^{(n)}(x)| \leq M^n$$

$\forall n=1, 2, \dots$. Then $\forall x \in (a-R, a+R)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Pf: $\forall x \in (a-R, a+R)$, $\exists c_x$ s.t

$$\begin{aligned} R_N(x; a) &= f(x) - T_N(x; a) \\ &= \frac{f^{(n+1)}(c_x)}{(n+1)!} (x-a)^{n+1} \end{aligned}$$

$$\text{So } |R_N(x; a)| \leq \frac{|f^{(n+1)}(c_x)|}{(n+1)!} |x-a|^{n+1}$$

$$\leq \frac{M^{n+1} \cdot R^{n+1}}{(n+1)!}$$

Claim: For any $p \in (0, \infty)$, $\lim_{n \rightarrow \infty} \frac{P^n}{n!} = 0$

Pf: $e^P = \sum_{n=0}^{\infty} P^n/n!$ converges

So divergence test $\Rightarrow \lim_{n \rightarrow \infty} \frac{P^n}{n!} = 0$.

Claim $\Rightarrow \lim_{N \rightarrow \infty} R_N(x; a) = 0$ & $f(x) = T_f(x; a)$.

Ex: Sps $E: \mathbb{R} \rightarrow \mathbb{R}$ s.t. $\begin{cases} E'(x) = E(x) \\ E(0) = 1 \end{cases}$

Claim: $E(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Pf: Clearly $E^{(n+1)}(x) = E(x)$. Let $R > 0$,

and $\sup_{(-R, R)} |E(x)| = M \geq 1$. Then

$$\sup_{(-R, R)} |E^{(n+1)}(x)| = M \leq M^n.$$

So Thm $\Rightarrow T_E(x; 0) = E(x)$ on $|x| < R$.

$$\text{But } T_E(x; 0) = \sum_{n=0}^{\infty} \frac{E^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{E(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Since this holds $\forall R > 0$:

$$E(x) = e^x \text{ on } (-\infty, \infty).$$

