

## 2.1 METRIC SPACES: Open & Closed Set

Def<sup>n</sup>: Let  $X$  be a set. A metric or distance function on  $X$  is a function

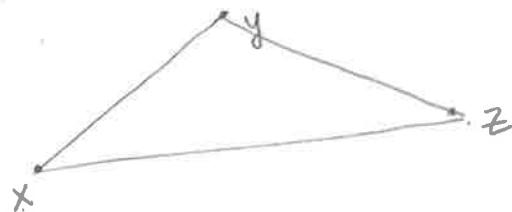
$$d: X \times X \rightarrow \mathbb{R} \\ (x, y) \mapsto d(x, y)$$

s.t

(1) (Positive definiteness)  $d(x, y) > 0$  if  $x \neq y$  and  
 $d(x, x) = 0$

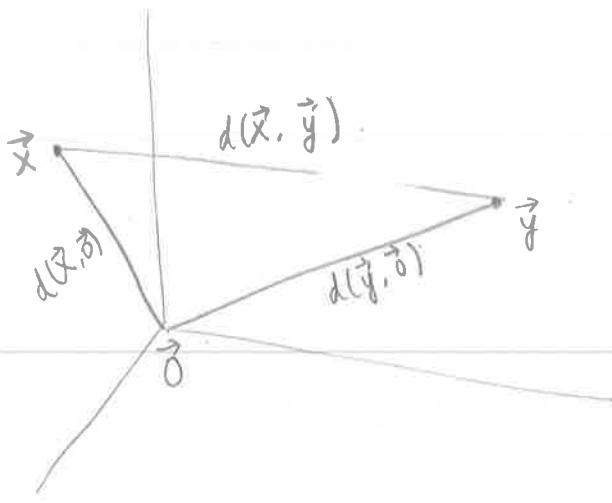
(2) (Symmetry)  $d(x, y) = d(y, x)$

(3) (Triangle inequality):  $\forall x, y, z \in X$   
 $d(x, y) \leq d(x, z) + d(y, z)$



We then call the pair  $(X, d)$  a metric space.  
The elements of  $X$  are sometimes referred to as points.

Examples: 1) Euclidean Spaces:  $\mathbb{R}^n$  For  $\vec{x}, \vec{y} \in \mathbb{R}^n$   
we let  $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$ .



Then clearly (i) & (ii) are satisfied. For  $\Delta$ -ineq  
thm i-ii applied to  $\vec{u} = \vec{x} - \vec{z}$ ,  $\vec{v} = \vec{y} - \vec{z}$

$$\begin{aligned}
 d(\vec{x}, \vec{y}) &= |\vec{x} - \vec{y}| = |\vec{u} - \vec{v}| \\
 &\leq |\vec{u}| + |\vec{v}| \\
 &= |\vec{x} - \vec{z}| + |\vec{y} - \vec{z}| \\
 &= d(\vec{x}, \vec{z}) + d(\vec{y}, \vec{z}).
 \end{aligned}$$

$\Rightarrow (\mathbb{C}, 1\cdot 1)$ . For  $z, w \in \mathbb{C}$ , we define

$$d(z, w) = |z - w|.$$

Then  $d$  is a metric on  $\mathbb{C}$ .

Note: If  $z = x + iy$ ,  $w = u + iv$ .

$$d(z, w) = \sqrt{(x-u)^2 + (y-v)^2}$$

So as metric spaces we can identify.

$(\mathbb{R}^2, 1\cdot 1)$  and  $(\mathbb{C}, 1\cdot 1)$ .

3). (Pathological Example). Let  $X$  be any set and define

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0 & x = y \end{cases}$$

Check:  $(X, d)$  is a metric space. (Assignment 2)

4)  $(\mathbb{Q}, 1 \cdot 1)$ . For any  $p, q \in \mathbb{Q}$  define

$$d(p, q) = |p - q|$$

Note that  $\mathbb{Q} \subset \mathbb{R}$  and the distance function is precisely the same as the usual Euclidean metric on  $\mathbb{R}$ . We then can call  $(\mathbb{Q}, 1 \cdot 1)$  a metric sub-space of  $\mathbb{R}$ .

Def<sup>n</sup>: Let  $(X, d)$  be a metric space. For any  $A \subset X$ , we can define a metric

$$d_A(x, y) = d(x, y)$$

$$\text{i.e } d_A = d|_{A \times A}$$

Then  $d_A$  defines a metric on  $A$ , and  $(A, d_A)$  is called a metric sub-space of

$(X, d)$

- Open & Closed Sets Let  $(X, d)$  be a metric space.

Def<sup>n</sup>: For  $p \in X$  and  $r > 0$ , the (open) ball of radius ' $r$ ' around  $p$  is defined to be

$$B_r(p) = \{x \in X \mid d(p, x) < r\}.$$

The closed ball of radius  $r$  is defined to be

$$B_{\leq r}(p) = \{x \in X \mid d(p, x) \leq r\}.$$

Def<sup>n</sup>: 1) A set  $U \subset X$  is called open if  $\forall p \in U, \exists r > 0$  (depending on  $p$ ) s.t

$$B_r(p) \subset U$$



2) A set  $A \subset X$  is called closed if  $A^c = X \setminus A$  is open.

Rk:  $X$  is always an open set in  $X$ . By convention,  $\emptyset$  (empty set) is also considered open.  
 $\Rightarrow \emptyset$  and  $X$  are both open & closed sets.

Examples: 1)  $(a, b)$  is open in  $\mathbb{R}$ ,  $[a, b]$  is closed in  $\mathbb{R}$ .  $(a, b]$  is neither.

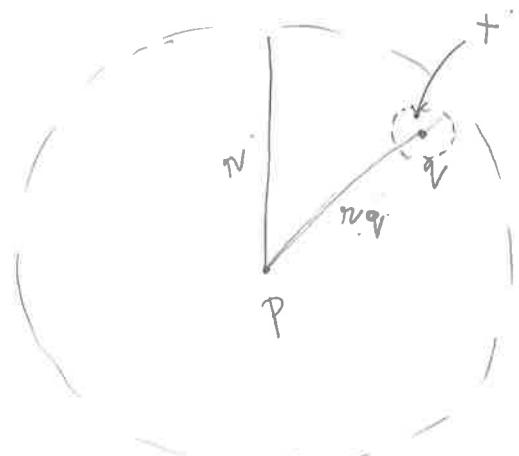
2)  $X$  with the pathological metric. Then every singleton set  $\{x\}$  is open & closed.

3) Set  $[\sqrt{2}, \sqrt{3}]$  is closed in  $\mathbb{R}$ , but  $[\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$  is both open & closed in  $\mathbb{Q}$  with the sub-space metric.

Prop 2.1.1 1) The ball  $B_r(p)$  is an open set in  $X$   $\forall p \in X, \forall r > 0$ .

2) The ball  $B_{\leq r}(p)$  is a closed set in  $X$   $\forall p \in X, \forall r > 0$ .

Pf: 1).



Let  $q \in B_r(p)$ . Then  $r_q = d(p, q) < r$ .

Let  $\varepsilon = \frac{r - r_q}{2}$ . (any  $\varepsilon < r - r_q$  would work)

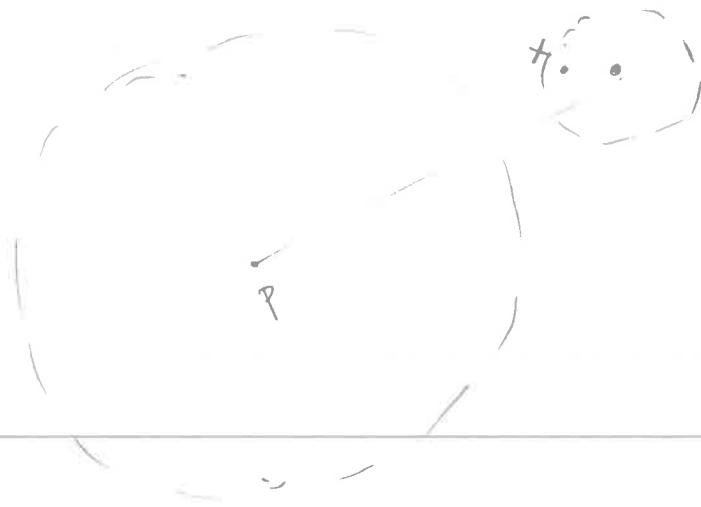
Claim:  $B_\varepsilon(q) \subset B_r(p)$ .

Pf: Let  $x \in B_\varepsilon(q)$ . Then

$$\begin{aligned} d(p, x) &\leq d(p, q) + d(x, q) < r_q + \varepsilon \\ &< r. \end{aligned}$$

$\Rightarrow x \in B_r(p)$ . So  $B_\varepsilon(q) \subset B_r(p)$ .

2). We show:  $X \setminus B_{\leq r}(p)$  is open.



Let  $q \in X \setminus B_{\leq r}(p)$ . Then  $r_q = d(p, q) > r$ .

Let  $\varepsilon = \frac{r_q - r}{2}$ .

Claim:  $B_\varepsilon(q) \subset X \setminus B_{\leq r}(p)$ .

Pf: Let  $x \in B_\varepsilon(q)$ . Sps.  $d(p, x) \leq r$ .

Then  $d(r_q) = d(p, q) \leq d(p, x) + d(q, x)$

$$\leq r + \varepsilon = \frac{r+r_2}{2} < r_2 \text{ since } r_2 > r.$$

So  $r_2 < r_2$  Contradiction !

$$\Rightarrow d(p, x) > r \Rightarrow x \in X \setminus B_{\leq r}(p)$$

Th<sup>m</sup> 2.1.2. 1) If  $\{G_\alpha\}$  is an arbitrary collection of open subsets of  $X$ , then

$$V = \bigcup_{\alpha \in I} G_\alpha$$

is also open in  $X$ .

2) If  $\{G_k\}_{k=1}^N$  is a finite collection of open subsets in  $X$ , then

$$W = \bigcap_{k=1}^N G_k$$

is open.

Rk: In 2)  $N$  has to be finite. Else consider

$$G_k = (-y_k, y_k).$$

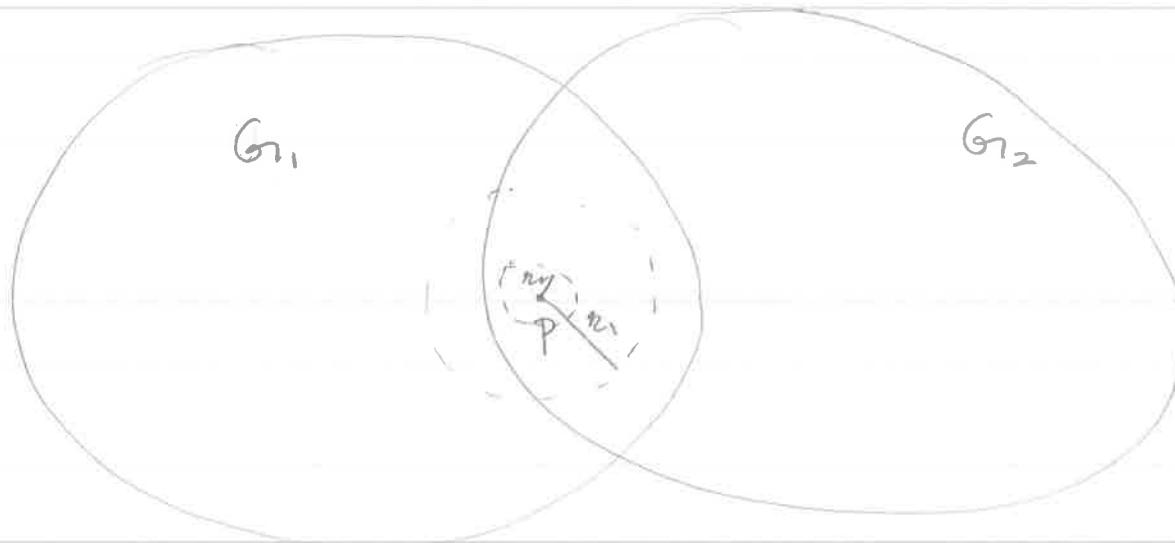
Then  $\bigcap_{k=1}^\infty G_k = \{0\}$  is NOT open.

Pf: i). let  $p \in V \Rightarrow p \in G_\alpha$  for some  $\alpha \in I$ .

$G_\alpha$  open  $\Rightarrow \exists r \text{ s.t. } B_r(p) \subset G_\alpha \subset V$ .

So  $V$  is open.

2).



let  $p \in W \Rightarrow p \in G_{r_k} \forall k = 1, 2, \dots, N$ .

$\forall k, \exists r_k > 0 \text{ s.t. } B_{r_k}(p) \subset G_{r_k}$

let  $r = \min(r_1, \dots, r_N)$  (here proof breaks down if  $N$  is NOT finite).

Then  $B_r(p) \subset B_{r_k}(p) \subset G_{r_k}$  since  $r < r_k$ .

$\Rightarrow B_r(p) \subset G_{r_k} \forall k \Rightarrow B_r(p) \subset W$ .

$\Rightarrow W$  is open!

Cor 2.1.3 i) If  $\{F_\alpha\}_{\alpha \in I}$  is an arbitrary collection of closed sets, then

$$A = \bigcap_{\alpha \in I} F_\alpha$$

is closed.

2) If  $\{F_k\}_{k=1}^N$  is a finite collection of closed sets, then

$$A = \bigcup_{k=1}^N F_k$$

is closed.

Pf: Follows from Thm 2.1.2 and identities

$$\left(\bigcap_{\alpha \in I} F_\alpha\right)^c = \bigcup_{\alpha \in I} F_\alpha^c$$

$$\left(\bigcup_{\alpha \in I} F_\alpha\right)^c = \bigcap_{\alpha \in I} F_\alpha^c$$

- Interior and limit points. Let  $(X, d)$  metric sp.

Defn: Let  $E \subset X$ .

i) A point  $p \in E$  is called an interior point of  $E$  if  $\exists r > 0$  s.t.  $B_r(p) \subset E$ .

2)  $p \in X$  is called a limit point if  $\forall r > 0$ ,

$\exists q \in B_r(p) \cap E$  s.t.  $q \neq p$ .

3)  $p \in E$  is called an isolated point if it is not a limit point i.e.  $\exists r > 0$  s.t.  
 $B_r(p) \cap E = \{p\}$ .

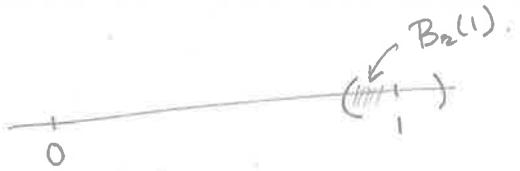
4) The set of all interior points of  $E$  is called the interior of  $E$ , denoted by  $E^\circ$ .

5) If  $E'$  is the set of all limit points of  $E$  then the closure of  $E$  is defined by  
 $\bar{E} = E \cup E'$ .

6) The boundary  $\partial E$  of  $E$  is defined as.

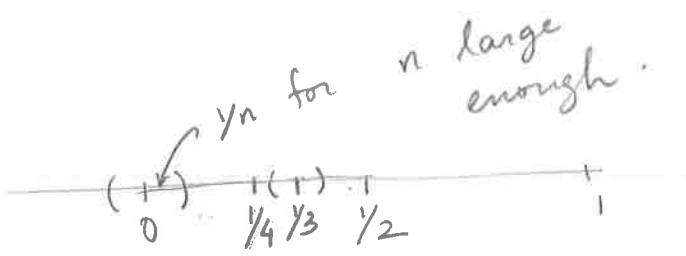
$$\partial E = \bar{E} \setminus E$$

Examples)  $X = \mathbb{R}$ ,  $E = [0, 1]$ . Any  $x \in (0, 1)$  is an interior point. So  $E^\circ = (0, 1)$ . 0 and 1 are l.p.



$$\text{So } \bar{E} = [0, 1], \quad \partial E = \{0, 1\}.$$

$$2) K = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$



Around any  $y_n$ , consider ball of radius

$$r_n = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2n(n+1)}.$$

$$B_{r_n}(y_n) \cap K = \left\{ \frac{1}{n} \right\}.$$

So  $y_n$  is an isolated point of  $K$ .

On the other hand  $0$  is a l.p. To see this, let  $r > 0$ . Then

$$B_r(0) \cap K = \left\{ \frac{1}{n} \mid n > \frac{1}{r} \right\} \neq \emptyset.$$

Thm 2.1.4: i) A set  $E$  is open  $\iff$

$$E = E^\circ.$$

ii) A set  $E$  is closed  $\iff$

$$E = \bar{E}.$$

Pf: (1) Obvious from definition

(2)  $\Rightarrow$  Clearly,  $E \subseteq \bar{E}$ . (from def<sup>n</sup> of  $\bar{E}$ )

To show reverse inclusion, let  $p \in \bar{E} \setminus E$ .

Since  $E$  is closed,  $X \setminus E$  is open. So  $\exists r$

s.t.  $B_r(p) \subset X \setminus E$ . But then  $B_r(p) \cap E$

$= \emptyset$ . So  $p$  cannot be a l.p of  $E$ .

Contradiction!

$\Leftarrow$  Now s.p.s  $\bar{E} = E$ . We show  $X \setminus E$ :

open. Let  $p \in X \setminus E$ . Since  $\bar{E} = E$ ,  $p \notin \bar{E}$

$\Rightarrow p$  is not a l.p of  $E$ . So  $\exists r$  s.t.

$\nexists q \in B_r(p) \cap E$ ,  $q \neq p$ . But  $p \notin B_r(p) \cap E$ .

$\Rightarrow B_r(p) \cap E = \emptyset$ . So all points of  $X \setminus E$  are interior points  $\Rightarrow X \setminus E$  is open.

Th<sup>m</sup> 2.1.5: Let  $E \subseteq X$ .

1) If  $U$  is any open set,  $U \subseteq E$ . Then,  $U \subseteq E^\circ$ . That is,  $E^\circ$  is the largest open set in  $E$ .

2) If  $A$  is a closed set &  $E \subseteq A$ .

Then  $\bar{E} \subseteq A$ . That is,  $\bar{E}$  is the smallest closed set containing  $E$ .

Pf: 1) Let  $p \in U$ .  $U$  open  $\Rightarrow \exists \varepsilon > 0$   
 $B_\varepsilon(p) \subseteq U \subseteq E \Rightarrow p \in E^\circ$ . So  $U \subseteq E^\circ$ .

2) Let  $p \in \bar{E}$ . If  $p \in E \Rightarrow p \in A$  since  $E \subset A$   
So sps.  $p \in \bar{E} \setminus E$ .

Claim:  $p \in A$ .

If not, then since  $X \setminus A$  is open,  $\exists \varepsilon > 0$   
 $B_\varepsilon(p) \cap A = \emptyset$ . But  $p$  is a l.p. of  $E$   
 $\Rightarrow B_\varepsilon(p) \cap E \neq \emptyset$  But since  $E \subset A$  this  
is a contradiction.

So  $p \in A$ , and hence  $\bar{E} \subset A$ .

Thm 2.1.6: If  $p$  is a l.p. of  $E$ , Then  $\forall \varepsilon$   
 $B_\varepsilon(p) \cap E$  contains infinitely many points

Pf: If not. That is  $\exists \varepsilon_0 > 0$  s.t.  $B_{\varepsilon_0}(p) \cap E$   
 $= \{x_1, \dots, x_N\}$ . Let  $\varepsilon < \min(d(p, x_1), \dots, d(p, x_N))$

Then  $B_\varepsilon(p) \cap E \subset B_{\varepsilon_0}(p) \cap E$  since  
 $\varepsilon_0 > \varepsilon$ . But  $x_k \notin B_\varepsilon(p) \cap E \quad \forall k = 1, \dots, N$ .

Contradiction.

## Dense Sets

Defn: A set  $E \subset X$  is called dense if

$$\overline{E} = X$$

Thm 2.1.7:  $\mathbb{Q} \subset \mathbb{R}$  is dense.

Pf: Let  $p \in \mathbb{R} \setminus \mathbb{Q}$  and  $r > 0$ .

$$(p-r, p, p+r)$$

Then by Thm 1.,  $\exists q \in \mathbb{Q}$  s.t

$$p-r < q < p$$

So  $B_r(p) \cap \mathbb{Q} \neq \emptyset$ .

$\Rightarrow p$  l.p. for  $\mathbb{Q}$ .