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3-limits & Continuity

• (X, d_X) and (Y, d_Y) metric spaces.

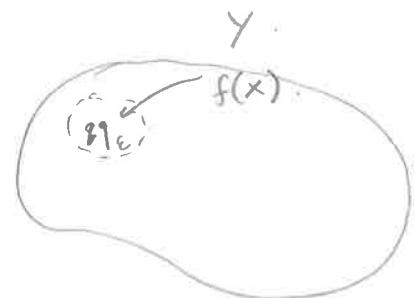
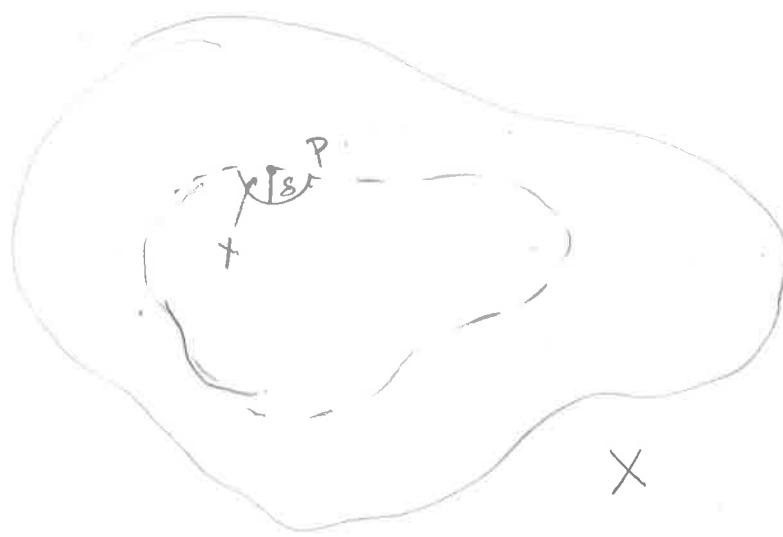
Defⁿ: Let $E \subset X$ and $f: E \rightarrow Y$. For $p \in X$ a l.p of E , we say.

$$\lim_{x \rightarrow p} f(x) = q$$

if $\forall \varepsilon > 0$, $\exists s = s(\varepsilon, p)$ s.t.

$$d_X(p, x) < s \Rightarrow d_Y(f(x), q) < \varepsilon$$

$x \in E, x \neq p$



Rk: 1) $f(p)$ need not be defined if $p \in \bar{E} \setminus E$

2) Even if $p \in E$, $f(p)$ might not be equal to $\lim_{x \rightarrow p} f(x)$

(2)

Example) $f: \mathbb{R} \rightarrow \mathbb{R}$

$$x \mapsto x^2$$

Claim: $\lim_{x \rightarrow 2} f(x) = 4$.

Pf: Given $\epsilon > 0$.

Aim: Find δ s.t. $|x-2| < \delta \Rightarrow |x^2 - 4| < \epsilon$.

Now $|x^2 - 4| = |x-2||x+2|$.



If $\delta < 1$, then $|x-2| < \delta \Rightarrow 1 < x < 3$
and so $|x+2| < 5$

So $|x^2 - 4| \leq 5|x-2|$ if $|x-2| < \delta$
 $\leq 5\delta$.

Now choose $\delta < \epsilon/5$.

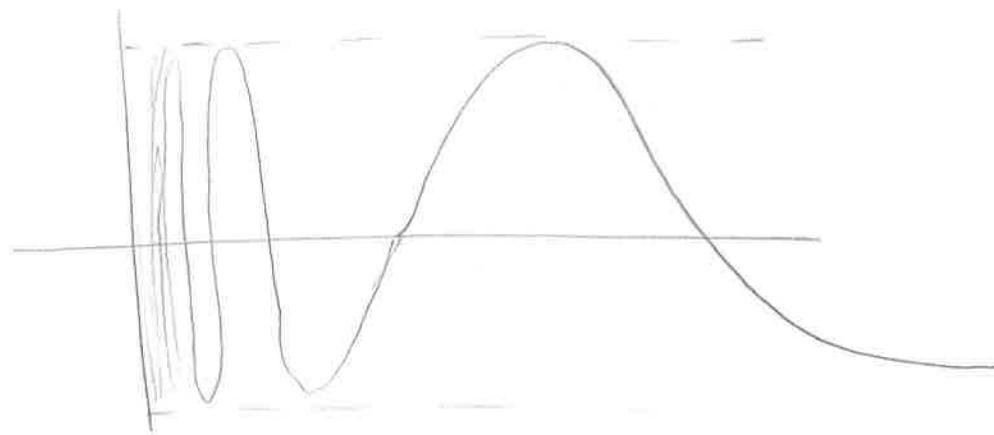
(3)

Then $|x^2 - 4| < 5\delta < \varepsilon$.

So if we let $\delta = \min(1, \varepsilon/5)$.

$$|x - 2| < \delta \Rightarrow |x^2 - 4| < \varepsilon.$$

2). $f: (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \sin(\gamma x)$



In this case $\lim_{x \rightarrow 0} f(x)$ DNE.

Thm 3.1: i) $\lim_{x \rightarrow p} f(x) = q \iff \forall P_n \rightarrow p, P_n \in E, P_n \neq p$

we have $\lim_{n \rightarrow \infty} f(P_n) = q$.

2) limits if they exist are unique.

Pf: i) \Rightarrow Let $\varepsilon > 0$, and $P_n \rightarrow p, P_n \in E$.

Then $\exists S$ s.t

$$\underset{x \in E}{d(p, x) < S \Rightarrow d_y(f(x), q) < \varepsilon}.$$

$P_n \rightarrow p \Rightarrow \exists N$ s.t

$$n > N \implies d_x(p_n, p) < \delta.$$

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$$\text{So } n > N \implies d_y(f(p_n), q) < \varepsilon.$$

$$\text{So } f(p_n) \rightarrow q.$$

\Leftarrow Sps. $\lim_{x \rightarrow p} f(x) \neq q$. Then $\exists \varepsilon > 0$

s.t. $\forall \delta$, $\exists x_s \in E$ s.t.

$$0 < d_x(x_s, p) < \delta \text{ but } d_y(f(x_s), q) > \varepsilon.$$

Let $\delta = y_n$, so $\exists \text{ seq } x_n \in E$ s.t.

$$0 < d_x(x_n, p) < y_n, \text{ but } d_y(f(x_n), q) > \varepsilon.$$

$$\text{So } x_n \rightarrow p, x_n \neq p.$$

$$\text{But } \lim_{n \rightarrow \infty} f(x_n) \neq q.$$

Contradiction

2). Follows from the fact that limits of sequences are unique.

Example: Again for $f(x) = \sin(\frac{1}{x})$.

$$\text{let } p_n = \frac{1}{n\pi}, q_n = \frac{2}{(2n+1)\pi}$$

Then $p_n \rightarrow 0, q_n \rightarrow 0$ but

$$f(p_n) = 0, f(q_n) = 1$$

So $\lim_{x \rightarrow 0} f(x)$ DNE.

(5)

Th^m 3.2. Sps. $E \subset X$, p.l.p of E .

1) If $f, g: E \rightarrow \mathbb{C}$ s.t.

$$\lim_{x \rightarrow p} f(x) = A, \quad \lim_{x \rightarrow p} g(x) = B$$

Then

$$(a) \lim_{x \rightarrow p} f(x) \pm g(x) = A \pm B$$

$$(b) \lim_{x \rightarrow p} (fg)(x) = AB$$

$$(c) \lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B} \text{ if } B \neq 0$$

2) If $\vec{F}, \vec{G}: E \rightarrow \mathbb{R}^k$, $\vec{F} = (f_1, \dots, f_k)$, $\lim_{x \rightarrow p} \vec{F} = \vec{A}$,
 $\vec{G} = (g_1, \dots, g_k)$, $\lim_{x \rightarrow p} \vec{G} = \vec{B}$

Then (a) holds. Moreover

$$(b') \lim_{x \rightarrow p} (\vec{F} \cdot \vec{G})(x) = \vec{A} \cdot \vec{B}$$

Recall $\vec{F} \cdot \vec{G}(x) = \sum_{j=1}^k f_j(x) g_j(x)$

• Continuous functions (X, d_X) and (Y, d_Y) metric.

Defⁿ: $E \subset X$ and $p \in E$. We say $f: E \rightarrow Y$ is continuous at p if $\forall \varepsilon > 0$, \exists $S = S(\varepsilon, p)$ s.t.

$$\underset{x \in E}{d_x(p, x) < \delta} \implies d_y(f(x), f(p)) < \varepsilon. \quad (6)$$

We say f is cont. on E if it is cont. at every $p \in E$

If f is not cont. at p it is called discontinuous.

Rk 1) For f to be cont. at p , it needs

to be defined at p .

2) By default a function is cont. at an isolated point in its domain

3). If $E \subset \mathbb{R}^k$, f is cont at $\vec{p} \iff$

$\forall \varepsilon > 0$, $\exists \delta$ s.t

$$|\vec{h}| < \delta \implies |f(\vec{p} + \vec{h}) - f(\vec{p})| < \varepsilon.$$

Thm 3.3 Let p be a l.p of E . Then the

follo. are eq

(1) f is cont. at p

(2) $\lim_{x \rightarrow p} f(x) = f(p)$

(3) $f(p_n) \rightarrow f(p) \quad \forall p_n \rightarrow p$

Example: 1) Absolute value: Consider. (7)

$$f: \mathbb{R}^k \rightarrow \mathbb{R}$$

$$f(\vec{x}) = |\vec{x}|.$$

Claim: f is cont on \mathbb{R}^k .

Pf: Uses the "reverse" triangle ineq i.e.

$$\vec{a}, \vec{b} \in \mathbb{R}^k, \quad |\vec{a} - \vec{b}| \geq |\vec{a}| - |\vec{b}|.$$

To see this, s.p.s $|\vec{a}| < |\vec{b}|$.



$$\text{Then } ||\vec{a}| - |\vec{b}|| = |\vec{b}| - |\vec{a}|.$$

$$= |\vec{b} - \vec{a} + \vec{a}| - |\vec{a}|.$$

$$\leq |\vec{b} - \vec{a}| + |\vec{a}| - |\vec{a}| = |\vec{b} - \vec{a}|.$$

Other case ($|\vec{a}| \geq |\vec{b}|$) similar.

Now let $\vec{p} \in \mathbb{R}^k, \epsilon > 0$.

$$|f(\vec{p} + \vec{h}) - f(\vec{p})| = ||\vec{p} + \vec{h}| - |\vec{p}||$$

$$\geq |\vec{p} + \vec{h} - \vec{p}| = |\vec{h}|.$$

Let $s = \epsilon$. Then

$$|\vec{h}| < s \Rightarrow |f(\vec{p} + \vec{h}) - f(\vec{p})| < \epsilon. \text{ Done!}$$

Rk: s is independent of \vec{p} in this example.

$$2). f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2. \quad (8)$$

Claim: f is cont. on \mathbb{R} .

Pf: Let $a \in \mathbb{R}, \epsilon > 0$:

$$|f(x) - f(a)| = |x^2 - a^2| = |x-a||x+a|.$$



If $\boxed{s < 1}$ and $|x-a| < s$, Then $|x|$

$$\begin{aligned} |x+a| &= |x-a+2a| \\ &\leq |x-a| + 2|a| \\ &\leq 1 + 2|a|. \end{aligned}$$

So if $s < 1 \wedge |x-a| < s$. Then

$$|f(x) - f(a)| = |x-a||x+a| < s \cdot (1 + 2|a|).$$

Pick $\boxed{s < \frac{\epsilon}{1+2|a|}}$

Then if $s = \min\left(1, \frac{\epsilon}{1+2|a|}\right)$:

$$|x-a| < s \Rightarrow |f(x) - f(a)| < \epsilon.$$

Rk: Here s depended on ϵ .

More generally, we have (9)

Th^m3.4:) If X is any metric space, and $p \in X$.

Then $f: X \rightarrow \mathbb{R}$ defined by

$$f(x) = d(p, x)$$

is cont.

2) Polynomials $f(x) = a_n x^n + \dots + a_0$ are
cont on \mathbb{R} .

Th^m3.5:) If $f, g: E \rightarrow \mathbb{R}$ cont. at p & $a, b \in \mathbb{R}$

Then $af \pm bg, f \cdot g$ are cont at p . If
 $g(p) \neq 0 \Rightarrow f/g$ cont. at p .

2) $\vec{F} = (F_1, \dots, F_k): E \rightarrow \mathbb{R}^k$ cont. $\Leftrightarrow F_j: E \rightarrow \mathbb{R}$

cont for $j=1, 2, \dots, k$.

3) $\vec{F}, \vec{G}: E \rightarrow \mathbb{R}^k$ cont. at $p \Rightarrow a\vec{F} + b\vec{G}$ cont
at $p \in E$ & $a, b \in \mathbb{R}$.

• Constructing New Cont. function from old

1) Restriction: Let $E \subset X$, $E' \subset E$ and $f: E \rightarrow Y$.

Define $f|_{E'}: E' \rightarrow Y$ by

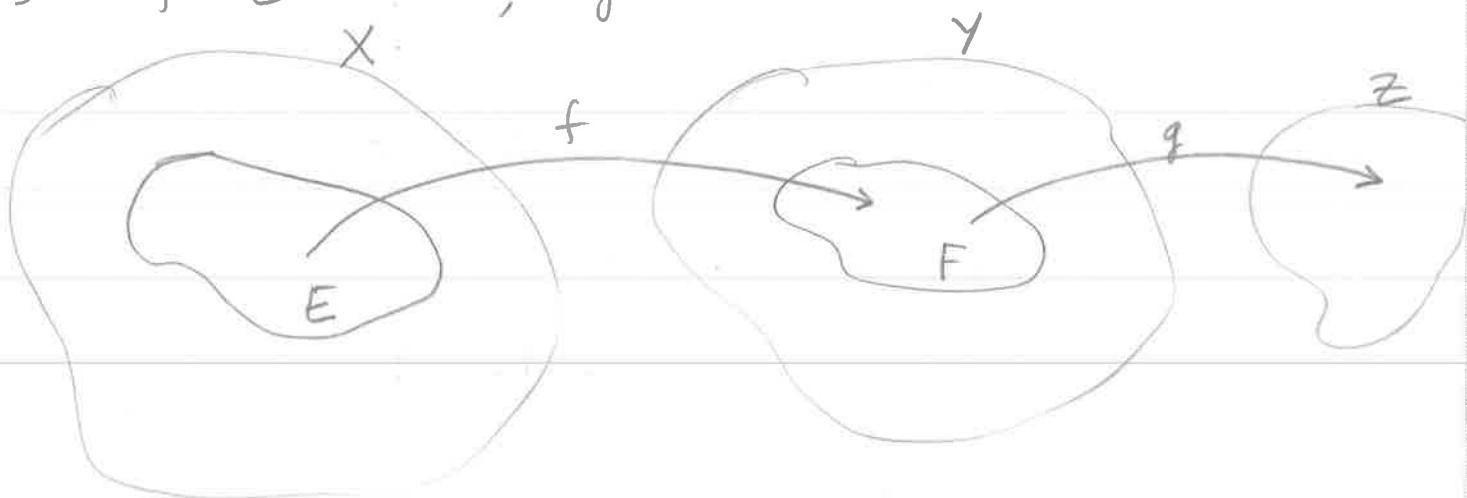
$$f|_{E'}(x) = f(x) \quad \forall x \in E'$$

Th^m3.6: If f is cont on E , then $f|_{E'}$ is
cont. on E' .

e.g.: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont. Then $f|_{(a,b)}: (a,b) \rightarrow \mathbb{R}$ is cont. for any $a < b$. (10)

2) Composition: Sp_c(X, d_X), (Y, d_Y) and (Z, d_Z) metric spaces and $E \subset X$, $F \subset Y$.

Sps f: E \rightarrow F; g: F \rightarrow Z.



Then

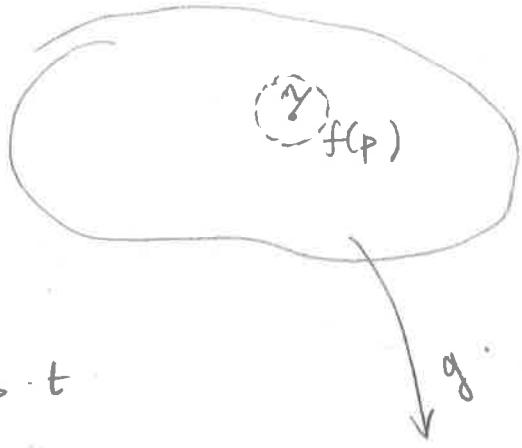
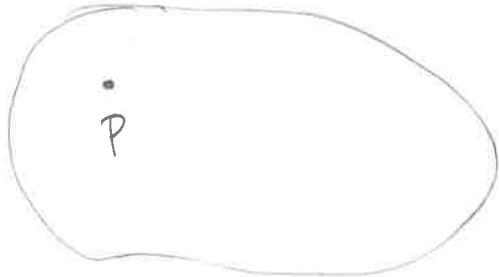
$$g \circ f: E \rightarrow Z.$$

defined by $g \circ f(x) = g(f(x))$.

Thm 3.7: If f, g, E & F as above. If f is cont. at $p \in E$ & g cont. at $f(p) \in F$. Then $g \circ f$ is cont. at $p \in E$.

Pf: Aim: Given $\epsilon > 0$, $\exists s$ s.t.

$$\forall x \in E, d_X(x, p) < s \implies d_Z(g \circ f(x), g(f(p))) < \epsilon.$$



g cont. $\Rightarrow \exists \eta > 0$ s.t.

$$d_y(y, f(p)) < \eta \Rightarrow d_z(g(y), g(f(p))) \\ < \varepsilon. (*)$$

f cont $\Rightarrow \exists \delta > 0$ s.t.

$$d_x(x, p) < \delta \Rightarrow d_y(f(x), f(p)) < \eta \quad (**)$$

Sps $d_x(x, p) < \delta \xrightarrow{**} d_y(f(x), f(p)) < \eta$.

$$\xrightarrow[\substack{(*) \\ y=f(x)}]{} d_z(g \circ f(x), g \circ f(p)) < \varepsilon. \text{ Done!}$$

Examples: $h(x) = \sqrt{x^2 + x^4}$ is cont on \mathbb{R}

Let $f(x) = x^2 + x^4$, $f: \mathbb{R} \rightarrow \mathbb{R}$
 $g(y) = \sqrt{y}$, $g: (0, \infty) \rightarrow \mathbb{R}$

f is cont by Th^m 3.4.

g is cont by Assignment problem
 $f(\mathbb{R}) \subset (0, \infty)$.

Th^m $\Rightarrow h \circ = g \circ f$ is cont.

Continuity & Open / Closed Sets

(12)

Th^m 3.8: Let $f: X \rightarrow Y$. Then foll. are equivalent

(1) f is cont. on X .

(2) For every open $V \subset Y$, $f^{-1}(V)$ is open in X .

(3) For every closed $E \subset Y$, $f^{-1}(E)$ is closed in X .

Recall, $f^{-1}(E) = \{x \in X \mid f(x) \in E\}$.

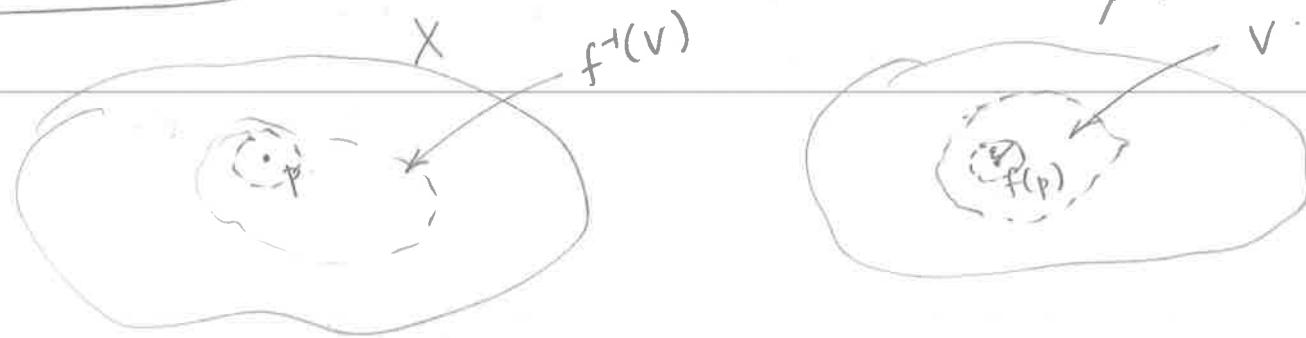
Rk: Image of open set need NOT be open.

e.g. Constant function. $f: \mathbb{R} \rightarrow \mathbb{R}$.

$$f(x) = 0$$

\mathbb{R} open, but $f(\mathbb{R}) = \{0\}$ not open.

Pf: $(1) \Rightarrow (2)$: Let $V \subset Y$ open.



Let $p \in f^{-1}(V) \Rightarrow f(p) \in V$

V open $\Rightarrow \exists \epsilon \text{ s.t. } B_\epsilon^Y(f(p)) \subset V$

Cont. $\Rightarrow \exists \delta > 0 \text{ s.t. }$

$d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \epsilon$.

That is,

$$f(B_s^x(p)) \subset B_\epsilon^y(f(p)) \subset V.$$

Taking f^{-1} , $B_s^x(p) \subset f^{-1}(V)$.

So p is an interior point of $f^{-1}(V)$.

$\Rightarrow f^{-1}(V)$ is open since this is true
for all $p \in f^{-1}(V)$.

(2) \Rightarrow (1). Let $\epsilon > 0$, $p \in X$.

Aim: $\exists s > 0$ s.t.

$$d_x(x, p) < s \Rightarrow d_y(f(x), f(p)) < \epsilon.$$

To see this, let $V = B_\epsilon^y(f(p))$.

(2) $\Rightarrow f^{-1}(V)$ is open

Clearly $p \in f^{-1}(V)$ since $f(p) \in V$

$f^{-1}(V)$ open $\Rightarrow p$ is an interior point

$\exists s$ s.t. $B_s^x(p) \subset f^{-1}(V)$.

If $d_x(x, p) < s$ (i.e. $x \in B_s^x(p)$) then

$x \in f^{-1}(V)$ or $f(x) \in V = B_\epsilon^y(f(p))$

$\Leftrightarrow d_y(f(x), f(p)) < \epsilon$. Done!

(14)

(2) \Leftrightarrow (3).

Claim: Since domain of f is all of X ,

$$f^{-1}(E^c) = [f^{-1}(E)]^c.$$

Pf: $x \in [f^{-1}(E)]^c \Leftrightarrow x \in X \setminus f^{-1}(E)$

$$\Leftrightarrow f(x) \notin E \quad (\text{This uses the fact that } f(x) \text{ is defined} \\ \nexists x \in X \setminus f^{-1}(E))$$

$$\Leftrightarrow f(x) \in E^c$$

$$\Leftrightarrow x \in f^{-1}(E^c).$$

Now, for (2) \Rightarrow (3), sps. $E \subset Y$ closed.

Then $V = E^c$ is open. So, (2) $\Rightarrow f^{-1}(V)$

is open. $\xrightarrow{\text{claim}} [f^{-1}(E)]^c$ is open.

$\Rightarrow f^{-1}(E)$ is closed.

For (3) \Rightarrow (2), sps $V \subset Y$ is open. Then

Letting $E = V^c$, E is closed.

(3) $\Rightarrow f^{-1}(E)$ is closed.

$\Rightarrow [f^{-1}(E)]^c$ is open.

$\xrightarrow{\text{claim}} f^{-1}(E^c)$ is open.

$\xrightarrow{E^c = V} f^{-1}(V)$ is open \Rightarrow (2).

• Continuity & Compact Sets $(X, d_X), (Y, d_Y)$ (15)

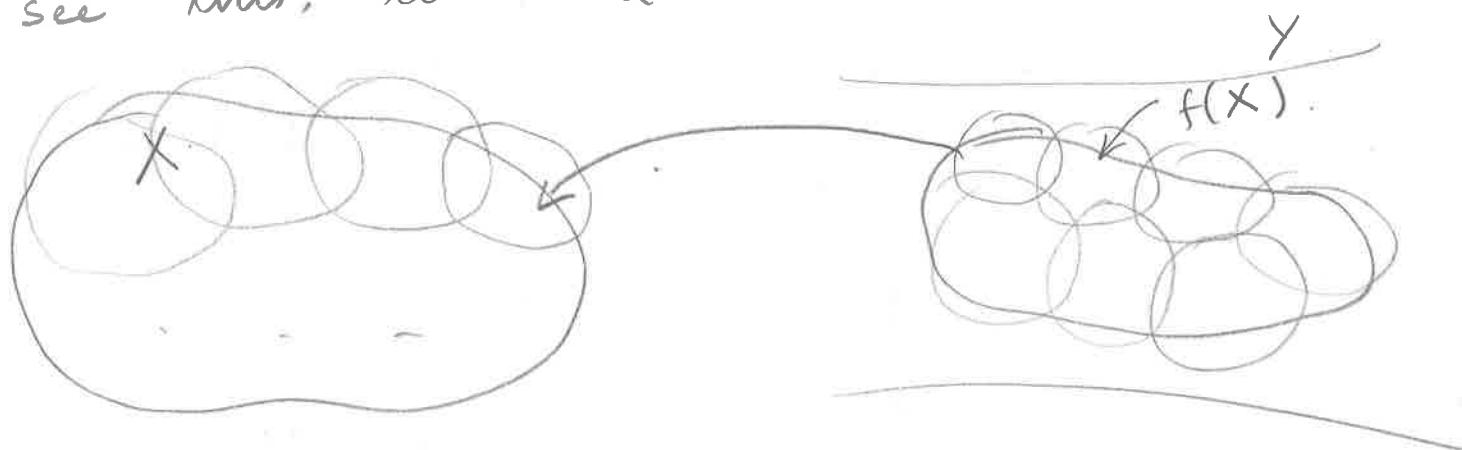
Th^m 3.9. Let $f: X \rightarrow Y$ continuous, and X is compact. Then $f(X)$ is compact.

Pf: Let $\{V_\alpha\}_{\alpha \in I}$ be any open cover of $f(X)$

Aim: $\exists \alpha_1, \dots, \alpha_N \in I$ s.t.

$$f(X) \subset \bigcup_{k=1}^N V_{\alpha_k}$$

To see this, let $U_\alpha = f^{-1}(V_\alpha)$.



Th^m 3.9 $\Rightarrow U_\alpha$ is open

Claim: $X \subset \bigcup_{\alpha \in I} U_\alpha$

Pf: $x \in X$, then $f(x) \in V_\beta$ for some $\beta \in I$
 $\Rightarrow x \in f^{-1}(V_\beta) = U_\beta$ for some $\beta \in I$
 $\Rightarrow x \in \bigcup_{\alpha \in I} U_\alpha$

So $\{U_\alpha\}_{\alpha \in I}$ is an open cover for X . (16)

X compact $\Rightarrow \exists \alpha_1, \dots, \alpha_N$ s.t

$$X \subset \bigcup_{k=1}^N U_{\alpha_k} \quad (*).$$

Claim: $f(X) \subset \bigcup_{k=1}^N V_{\alpha_k}$

Pf: Let $y \in f(X)$. Then $\exists x \in X$ s.t

$$\begin{aligned} y &= f(x), & (\forall) \\ (*) \rightarrow x &\in U_{\alpha_k} \text{ for some } k \in \{1, \dots, N\} \\ \Rightarrow x &\in f^{-1}(V_{\alpha_k}) \\ \Rightarrow y &= f(x) \in V_{\alpha_k}. \end{aligned}$$

Done!

So $\{V_{\alpha_k}\}_{k=1}^N$ is a finite sub-cover.

Defⁿ: A function $f: X \rightarrow Y$ is called bounded if $f(X)$ is a bounded set.

Cor 3.10: If X is compact, then any continuous $f: X \rightarrow Y$ is bounded.

Pf: $f(X)$ is compact & so bounded.

Cor 3.11 (Extremum value) Let X be compact & $f: X \rightarrow \mathbb{R}$ continuous, then

$$M = \sup_{x \in X} f(x), \quad m = \inf_{x \in X} f(x).$$

exist, and are attained i.e. $\exists p, q \in X$ s.t.

$$M = f(p), \quad m = f(q).$$

Pf: By Cor 3.10, $f(X)$ is bounded & so \sup & \inf exist. Let

$$M = \sup_{x \in X} f(x), \quad m = \inf_{x \in X} f(x).$$

Claim: $M \in \overline{f(X)}$

Pf: If not, then $\exists r > 0$ s.t.

$$\overbrace{\quad\quad\quad}^{(M-r, M, M+r)}.$$

$$(M-r, M+r) \cap f(X) = \emptyset.$$

$$\text{Since } f(x) \leq M \quad \forall x \in X$$

$$\Rightarrow f(x) \leq M-r \quad \forall x \in X$$

$$\Rightarrow M-r \text{ is another u.b.}$$

Contradiction.

But $f(X)$ compact $\Rightarrow \overline{f(X)} = f(X)$ (18)

$\Rightarrow M \in f(X) \Rightarrow \exists p \in X \text{ s.t. } f(p) = M$

Similarly $m \in f(X) \& \exists q \in X \text{ s.t. } f(q) = m$.

Rk: Often applied with $X = [a, b] \subset \mathbb{R}$

Defⁿ: With notation as above, p is called the maxima^(max) & q is called the minima^(min). M & m are called the max & min values. Together they are called extremas & extremum values resp.

Example: Consider $f: [0, \infty) \rightarrow \mathbb{R}$

$$f(x) = \frac{1}{1+x^2}.$$

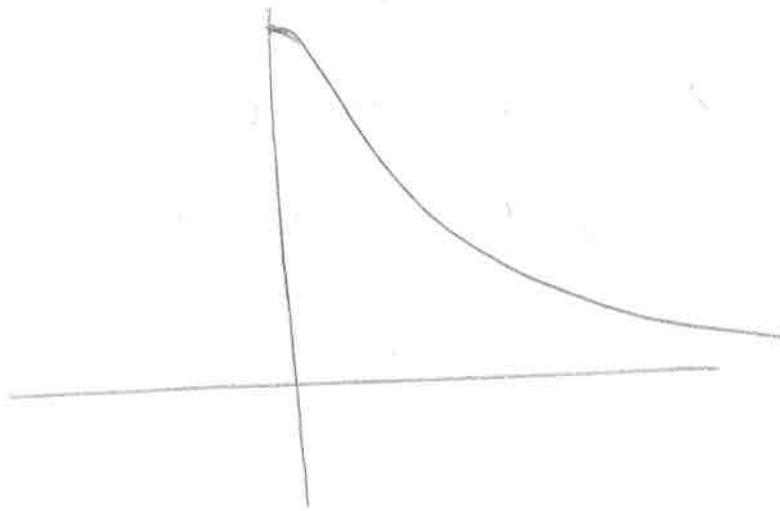
Clearly, since $x^2 \geq 0, \frac{1}{1+x^2} \leq 1$.

Also $f(0) = 1$.

So 0 is a max & max value is 1.

Also, $f(n) = \frac{1}{1+n^2} \xrightarrow{n \rightarrow \infty} 0$

But $f(x) \neq 0 \forall x$.



So, f has no min on \mathbb{R} .

But, by theorem, on any compact $[0, a]$.

f has a minima, namely $x = a$.

Similarly, no minima on $[0, a)$.

Continuity & Connected Sets

Th^m 3.12 Let $f: X \rightarrow Y$ be cont. Sps. ~~ECX~~ is connected, then $f(E)$ is also connected.

Pf: Sps not. Then

$$f(E) = A \cup B, \quad A, B \neq \emptyset$$

$$\overline{A}^Y \cap B = \overline{B}^Y \cap A = \emptyset$$

Let $G_1 = f^{-1}(A) \cap E$, $G_1, H \neq \emptyset$.

$H = f^{-1}(B) \cap E$.

Claim 1: $E = G \cup H$.

Pf: Clearly $G \cup H \subseteq E$. Sps $x \in E$. Then (20)

$$f(x) \in f(E) = A \cup B.$$

$$\Rightarrow f(x) \in A \text{ or } B.$$

$$\Rightarrow x \in f^{-1}(A) \text{ or } f^{-1}(B).$$

$$\Rightarrow x \in f^{-1}(A) \cap E \text{ or } f^{-1}(B) \cap E.$$

Claim 2: $\bar{G} \cap H = \bar{H} \cap G = \emptyset$.

Pf: Sps $x \in \bar{G} \cap H \Rightarrow f(x) \in B$.

CASE 1: $x \in G$. Then $f(x) \in A$ and so
 $f(x) \in A \cap B$ contradiction!

CASE 2: $x \in \bar{G} \setminus G$. Then $\exists x_n \in G$ s.t
 $x_n \rightarrow x$.

f cont. $\Rightarrow f(x_n) \rightarrow f(x)$.

$f(x_n) \in A \quad \forall n$.

$\Rightarrow f(x)$ is a l.p. of A

$\Rightarrow f(x) \in \bar{A}$

$\Rightarrow f(x) \in \bar{A} \cap B$ Contradiction

So $\bar{G} \cap H = \emptyset$. Similarly $\bar{H} \cap G = \emptyset$.

So $E = G \cup H, \quad G, H \neq \emptyset$
 $G \cap \bar{H} = H \cap \bar{G} = \emptyset$

$\Rightarrow E$ disconnected. Contradiction

Cor 3.13 (Intermediate value Th^m) Let $E \subset \mathbb{R}$,⁽²¹⁾
 and $f: E \rightarrow \mathbb{R}$ cont. Let $[a, b] \subset E$ and c be between
 $f(a), f(b)$. Then $\exists x \in [a, b]$ s.t
 $f(x) = c$. (See figure).

Pf: $f([a, b])$ is connected in \mathbb{R} by Th^m 3.12
 Sps $f(a) \leq f(b)$, then the entire interval
 $[f(a), f(b)] \subset f([a, b]) \Rightarrow \forall c \in [f(a), f(b)]$.
 $\exists x \in [a, b]$ s.t $f(x) = c$.

Example: Any cubic

$$p(x) = x^3 + px + q$$

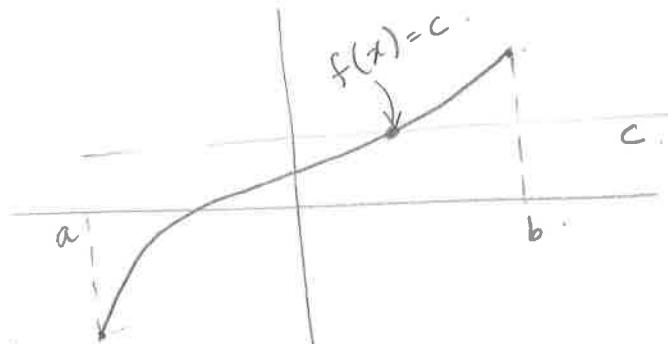
has at least one real root.

To see this, note as $x \rightarrow -\infty$, $p(x) \rightarrow -\infty$.
 i.e. If a is really -ve., then

$$p(a) < 0$$

Also: $p(x) \rightarrow +\infty$ as $x \rightarrow +\infty$. So $\exists b > 0$ s.t
 $p(b) > 0$.

Cor $\stackrel{(c=0)}{\Rightarrow} \exists x$ s.t $p(x) = 0$.



Uniform Continuity

Examples: 1) $f(x) = ax + b$. Given $\epsilon > 0$,

$$|f(x) - f(p)| = a|x - p| < \alpha s$$

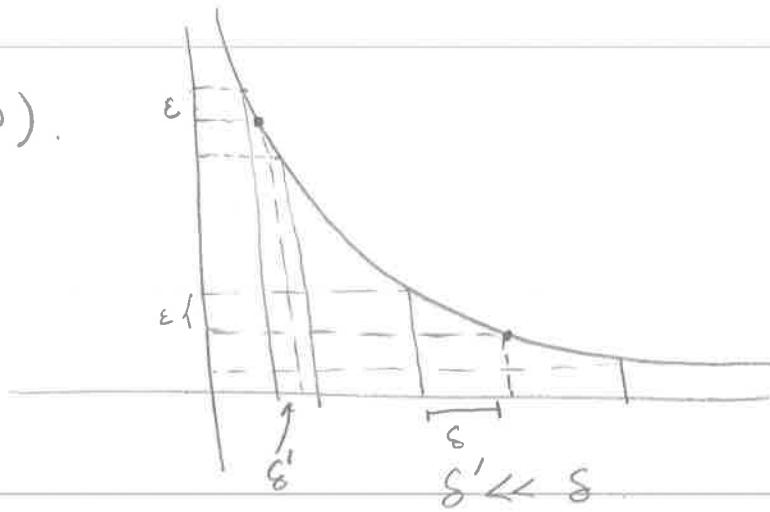
Pick $s = \epsilon/a$. So s independent of p .

2) $f(x) = x^2$. We saw earlier that to prove cont. at $x = a$, $s = \min\left(1, \frac{\epsilon}{1+2|a|}\right)$

and depends on a .

3) $f(x) = 1/x$.

cont. on $(0, \infty)$.



So s becomes smaller as you approach 0

Defⁿ: $f: X \rightarrow Y$ is said to be uniformly continuous if $\forall \epsilon > 0$, $\exists s = s(\epsilon)$ s.t. $\forall p \in X$.

$$d_X(x, p) < s \implies d_Y(f(x), f(p)) < \epsilon.$$

Rk: So δ can be chosen independent of p .

Example: $f(x)$ is not uniformly cont. on $(0, \infty)$. If it is, then $\forall \epsilon > 0$, $\exists \delta$

$$\text{s.t } |x - p| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{p} \right| < \epsilon.$$

$$\text{i.e. } \left| \frac{1}{x} - \frac{1}{p} \right| = \frac{|x-p|}{|xp|} < \epsilon.$$

In particular $\exists \delta$ s.t

$$|x - p| < \delta \Rightarrow \frac{|x-p|}{|xp|} < 1. \quad \forall p \in (0, \infty).$$

Take $p_n = 1/n$, $x_n = p_n + \frac{\delta}{2}$. So $|x_n - p_n| = \frac{\delta}{2} < \delta$

$$\text{But } \frac{|x_n - p_n|}{|x_n p_n|} = \frac{\frac{\delta}{2}}{\frac{1}{n} \left(\frac{1}{n} + \frac{\delta}{2} \right)} \xrightarrow{n \rightarrow \infty} \infty$$

Contradiction

Exercise: Show $f(x)$ is uniformly cont on $[a, \infty)$ $\forall a > 0$.

Thm 3.14: Sps $f: X \rightarrow Y$ is cont. If X is compact, then f is uniformly continuous.

Pf: Given $\varepsilon > 0$,

f cont. $\Rightarrow \forall p, \exists s_p > 0$ s.t.

$$d_X(x, p) < s_p \Rightarrow d_Y(f(x), f(p)) < \varepsilon/2 \quad (*)$$

Now,

$$X \subset \bigcup_{p \in X} B_{s_p/2}(p)$$

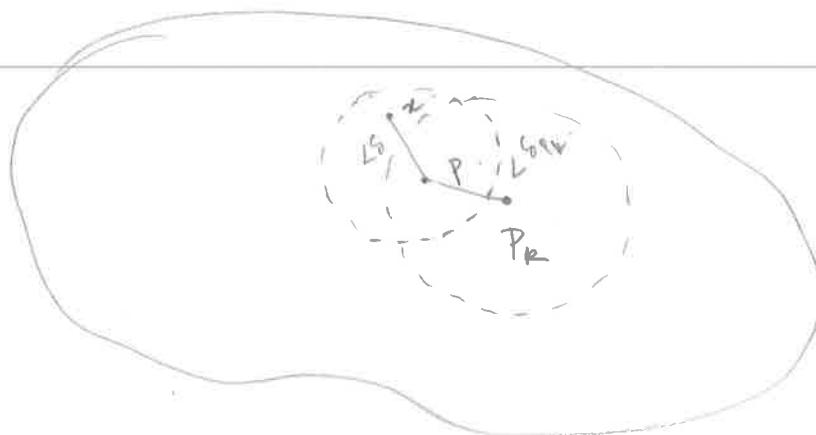
X compact $\Rightarrow \exists p_1, \dots, p_N \in X$ s.t

$$X \subset \bigcup_{k=1}^N B_{s_{p_k}/2}(p_k)$$

$$\text{let } S = \frac{1}{2} \min(s_{p_1}, s_{p_2}, \dots, s_{p_N})$$

Claim: $\forall p \in X, d_X(x, p) < S \Rightarrow d_Y(f(x), f(p)) < \varepsilon$.

Pf: $p \in B_{s_{p_k}/2}(p_k)$ for some p_k .



$$(*) \Rightarrow d_Y(f(p), f(p_k)) < \varepsilon/2$$

$$\text{Also } d_X(x, p) < S \Rightarrow$$

$$d_X(x, p_k) < d_X(x, p) + d_X(p, p_k)$$

$$< \delta + \frac{\delta_{p_k}}{2} < \frac{\delta_{p_k}}{2} + \frac{\delta_{p_k}}{2} = \delta_{p_k}$$

$$(*) \Rightarrow d_Y(f(x), f(p_k)) < \varepsilon/2.$$

$$\Rightarrow d_Y(f(x), f(p)) < d_Y(f(x), f(p_k)) + d_Y(f(p), f(p_k)) \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \text{ Done!}$$

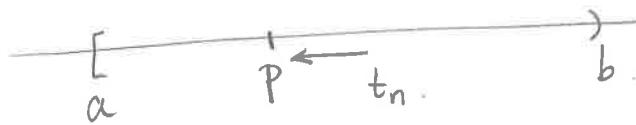
Some applications of uniform continuity will be in the assignment.

Discontinuities

Defⁿ: Let $f: (a, b) \rightarrow Y$. For $p \in (a, b)$, we say

$$f(p+) = q.$$

and say f has a right side limit q . if $f(t_n) \rightarrow q$ for any seqⁿ $t_n \rightarrow p$ with $t_n > p$.



We also write this as

$$\lim_{x \rightarrow p^+} f(x) = q.$$

We can similarly define $f(p-)$ or $\lim_{x \rightarrow p^-} f(x)$.

Rk: i) $f(p+) = q \Leftrightarrow \forall \varepsilon > 0, \exists \delta \text{ s.t.}$

$$p < x < p + \delta \Rightarrow |f(x) - q| < \varepsilon.$$

$f(p-) = q \Leftrightarrow \forall \varepsilon > 0, \exists \delta \text{ s.t.}$

$$p - \delta < x < p \Rightarrow |f(x) - q| < \varepsilon.$$

2) $\lim_{x \rightarrow p} f(x) = q \Leftrightarrow f(p+) = f(p-) = q.$

Defn: We say $f : (a, b) \rightarrow Y$ has a discont.
of the first kind at $p \in (a, b)$, if $f(p+)$ & $f(p-)$
exist BUT f is discont. at p .
Else we say f has discont. of second
kind at p .

Rk: f can have discont. of 1st kind if
any of the foll. two situations occur.

$$(a) f(p+) \neq f(p-)$$

$$(b) f(p+) = f(p-) \neq f(p).$$

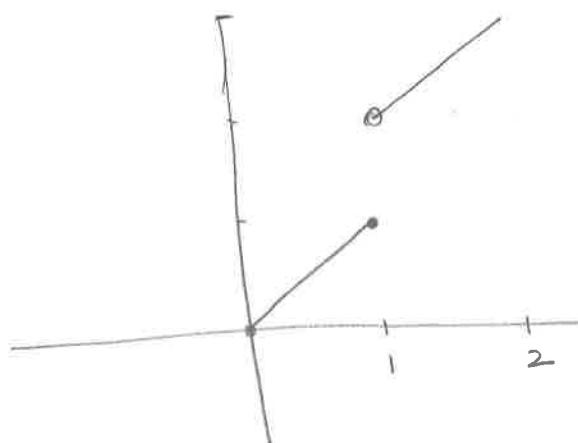
Example 1) $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Neither $f(p+)$ nor $f(p-)$ exist for any $p \in \mathbb{R}$.
 So discontin. of 2nd kind at all $p \in \mathbb{R}$.

2) $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

f is cont. at $p=0$, but has discontin. of 2nd kind at every $p \neq 0$.

3) $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x+1 & 1 < x < 2 \end{cases}$



Then f is cont. on $[0, 1) \cup (1, 2]$.

$$\text{At } p=1, \quad f(1-) = 1$$

$$f(1+) = 2.$$

So f has discontin. of 1st type at $p=1$.

There is one class of function (of which 3) is an example) which can only have discontinuities of 1st kind.

Defⁿ. (Monotonic functions) Let $f: (a, b) \rightarrow \mathbb{R}$

1) f is called increasing ($f \uparrow$) if

$$s < t \Rightarrow f(s) \leq f(t).$$

2) f is called decreasing ($f \downarrow$) if

$$s < t \Rightarrow f(s) \geq f(t).$$

f is called monotonic if it is increasing or decreasing.

Th^m 3.15 A monotonic function has no discontinuity of 2nd type i.e $f(p-)$ & $f(p+)$ always exist.

Rk) $f \uparrow \Rightarrow f(p-) \leq f(p+)$
 $f \downarrow \Rightarrow f(p-) \geq f(p+)$.

2) This is analogous to the theorem that bdd monotonic sequences have limits.

Pf: Sps $f \uparrow$. (Other case is similar).

For $p \in (a, b)$, let

$$A = \sup_{t \in [a, p]} f(t), \quad B = \inf_{t \in (p, b)} f(t).$$

$$f \uparrow \Rightarrow A \leq f(p) \leq B.$$

Claim: $f(p-) = A \quad \& \quad f(p+) = B$.

Pf: Let $t_n \rightarrow p$, $t_n \in (a, p)$.

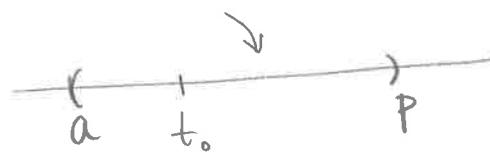


Clearly $f(t_n) \leq A$

Since $A = \sup_{(a, p)} f(t) \Rightarrow \forall \varepsilon > 0, \exists t_0 \in (a, p)$

$$\text{s.t. } f(t_0) > A - \varepsilon$$

else $A - \varepsilon$ would be an upper bound.



$$t_n \rightarrow p \Rightarrow \exists N \text{ s.t. } \forall n > N, f(t_n) > t_0.$$

(30)

$$f \uparrow \Rightarrow f(t_n) \geq f(t_0) > A - \varepsilon.$$

So $\forall n \geq N$

$$A - \varepsilon < f(t_n) \leq A$$

$$\text{or } |f(t_n) - A| < \varepsilon.$$

So $f(t_n) \rightarrow A$.

Hence $f(p-) = A$.

Similarly $f(p+) = B$.

This proves the theorem.