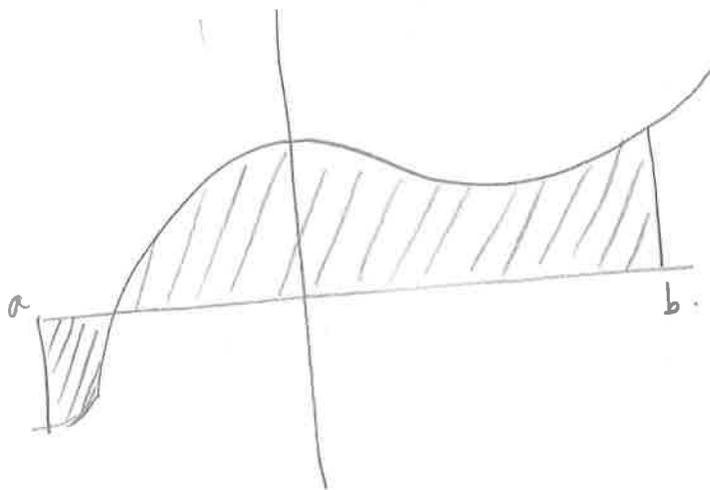


5. INTEGRATION

①

Aim: Given $f: [a, b] \rightarrow \mathbb{R}$, compute the (signed) area under the graph $y = f(x)$.

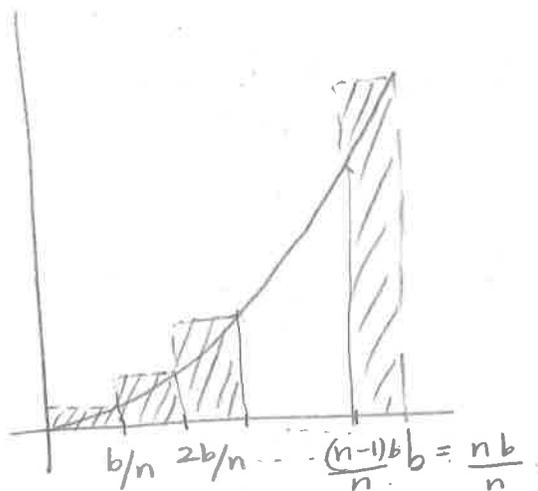


• Examples

1) $f(x) = x^2$ on $[0, b]$

Naive Idea: Right side approx.

let $n \in \mathbb{N}$,



Consider partition

$$0 = t_0 < t_1 = \frac{b}{n} < t_2 = \frac{2b}{n} < \dots < t_{n-1} = \frac{(n-1)b}{b} < t_n = b$$

If $A_{k,n}$ is the area of the k^{th} rectangle. ⁽²⁾

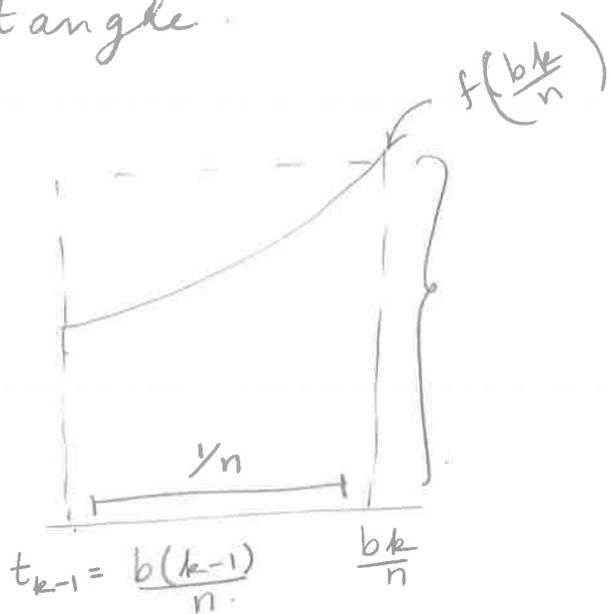
A_n is the sum of areas of rectangles.

$A =$ area under $y = x^2$ from 0 to b .

Intuitively,

$$A = \lim_{n \rightarrow \infty} A_n$$

The k^{th} rectangle.



$$\text{So } A_k = \frac{b^2 k^2}{n^2} \cdot \frac{1}{n} = \frac{b^2 k^2}{n^3}$$

$$\Rightarrow A_n = \sum_{k=1}^n A_k = \frac{b^2}{n^3} \sum_{k=1}^n k^2$$

FACT:

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow A_n = \frac{b^2}{n^2} \cdot \frac{(n+1)(2n+1)}{6}$$

$$= \frac{b^2}{6} (1 + \frac{1}{n})(2 + \frac{1}{n}) \rightarrow \frac{2b^2}{6} = \boxed{\frac{b^2}{3}}$$

So $A = \boxed{b^2/3}$

We write this as $\int_0^b t^2 dt = \frac{b^2}{3}$

2) Now consider

$$f(x) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$$

If we compute $\int_0^1 f(t) dt$ using right-hand rule. Consider $0 = t_0 < t_1 = 1/n \dots < t_n = 1$.

Then $f(t_n) = 1$, width = $1/n$.

So $A_n = \frac{1}{n} \cdot n \leftarrow \# \text{ of rectangles}$

$$\Rightarrow \int_0^1 f(t) dt = 1$$

For $\int_0^{\sqrt{2}} f(t) dt$, consider $t_0 = 0, t_1 = \sqrt{2}/n, \dots, t_n = \sqrt{2}$
 $f(t_k) = 0 \forall k$

$$\text{So } \int_0^{\sqrt{2}} f(t) dt = 0.$$

(4)

But $f(t) \geq 0$, so area under graph from 0 to 1 should be smaller than area from 0 to $\sqrt{2}$.

So this is a problem with right hand rule.

Riemann Integrability

Defⁿ: let $[a, b] \subset \mathbb{R}$, By a partition P we mean a finite set $\{t_0, \dots, t_n\}$.

$$a \leq t_0 < t_1 < \dots < t_{n-1} < t_n = b.$$

$$I_k = [t_{k-1}, t_k]$$



Write $\Delta t_k = t_k - t_{k-1}$

Sps $f: [a, b] \rightarrow \mathbb{R}$ is bounded, set

$$M_k = \sup_{t \in I_k} f(t)$$

$$m_k = \inf_{t \in I_k} f(t).$$

Define the upper and lower sums ⑤
by

$$U(P, f) := \sum_{k=1}^n M_k \Delta t_k$$

$$L(P, f) = \sum_{k=1}^n m_k \Delta t_k$$

If $M = \sup_{[a,b]} f$, $m = \inf_{f \in [a,b]} f$, then

$$m(b-a) < L(P, f) < U(P, f) < M(b-a)$$

for any P .

Defⁿ: We define the upper Riemann integral
and lower Riemann integral, by

$$\overline{\int_a^b} f(t) dt := \inf_P U(P, f)$$

$$\underline{\int_a^b} f(t) dt := \sup_P L(P, f)$$

We say f is Riemann integrable if both
are equal & then we set the integral as

$$\int_a^b f(t) dt := \overline{\int_a^b} f(t) dt = \underline{\int_a^b} f(t) dt$$

Let $R[a, b]$ = set of Riem. int. functions on $[a, b]$

Example: Again consider

⑥

$$f(x) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0 & t \notin \mathbb{Q}. \end{cases}$$

let P any partition $0 = t_0 < t_1 < \dots < t_n = 1$.

Then $U(P, f) = 1$, $L(P, f) = 0$.

$$\text{So } \int_0^1 f(t) dt = 1, \quad \int_0^1 f(t) dt = 0.$$

$\Rightarrow f$ is NOT Riemann integrable.

Criteria for integrability

Defⁿ: For a partition $P = \{t_0 = a < t_1 < \dots < t_n = b\}$

define $|P| = \max_{k=1, \dots, n} \Delta t_k = \max (t_k - t_{k-1})$.

Th^m 5.1 let $f: [a, b] \rightarrow \mathbb{R}$ be bounded. The foll are equivalent

(1) $f \in \mathcal{R}[a, b]$.

(2) $\forall \epsilon > 0, \exists P$ s.t.

$$U(P, f) - L(P, f) < \epsilon.$$

(3) $\forall \epsilon > 0, \exists \delta > 0$ s.t.

(7)

$$|P| < \delta \Rightarrow U(P, f) - L(P, f) < \epsilon.$$

Before proving this we need some preparation

Defⁿ: Given a partition P , another partition P^* is called a refinement if $P \subseteq P^*$

Lemma 5.2: If P^* is a refinement of P ,

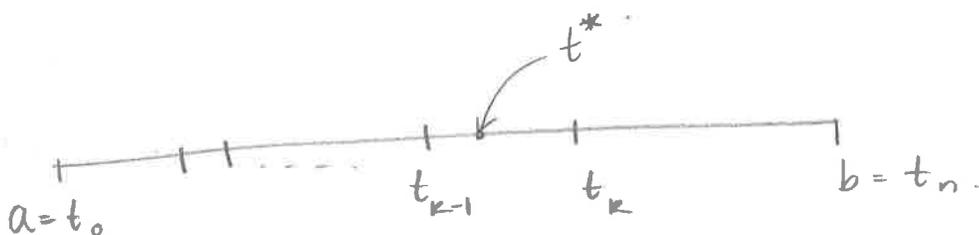
then

$$L(P, f) \underset{(1)}{\leq} L(P^*, f) \leq U(P^*, f) \underset{(2)}{\leq} U(P, f).$$

i.e. P^* gives a tighter bound on upper & lower integrals.

Pf: let's prove inequality (1). Pf of (2) is similar.

let $P = \{t_0, \dots, t_n\}$, Sps P^* has one extra point $t^* \in (t_{k-1}, t_k)$.



Like usual, let $m_j = \inf_{[t_{j-1}, t_j]} f(t)$.

(8)

Also denote

$$W_1 = \inf_{[t_{k-1}, t^*]} f(t), \quad W_2 = \inf_{[t^*, t_k]} f(t)$$

Key point: $W_1, W_2 \geq m_k$.

Now

$$L(P^*, f) = \sum_{j=1}^{k-1} m_j \Delta t_j + \sum_{j=k+1}^n m_j \Delta t_j$$

$$+ W_1 (t^* - t_{k-1}) + W_2 (t_k - t^*)$$

$$L(P, f) = \sum_{j=1}^n m_j \Delta t_j$$

So

$$L(P^*, f) - L(P, f) = W_1 (t^* - t_{k-1}) + W_2 (t_k - t^*) - m_k (t_k - t_{k-1})$$

$$\geq m_k (t^* - t_{k-1}) + m_k (t_k - t^*) - m_k (t_k - t_{k-1}) \geq 0$$

If P^* has more extra points, repeat this proof.

Proof of Theorem

(9)

(1) \Rightarrow (2). \therefore Sp. $f \in R[a, b]$. So

$$\sup_P L(P, f) = \inf_P U(P, f) = A = \int_a^b f(t) dt.$$

Let $\varepsilon > 0$. Then $\exists P_1$ and P_2 s.t.

$$U(P_1, f) - A < \frac{\varepsilon}{2} \quad (*)$$

$$A - L(P_2, f) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$. Then P is a refinement of P_1 and P_2 . So Lemma \Rightarrow

$$U(P, f) < U(P_1, f).$$

$$L(P, f) > L(P_2, f).$$

$(*) \Rightarrow$

$$U(P, f) - A < \varepsilon/2$$

$$A - L(P, f) < \varepsilon/2$$

Adding

$$U(P, f) - L(P, f) < \varepsilon/2.$$

(2) \Rightarrow (3). Let $\varepsilon > 0$. Then $\exists P^\varepsilon$ s.t. (10)

$$U(P^\varepsilon, f) - L(P^\varepsilon, f) < \frac{\varepsilon}{3}. \quad (**)$$

Let $n^\varepsilon = \#$ of partition points in P^ε i.e.

$$P^\varepsilon = \{t_0^\varepsilon, \dots, t_{n^\varepsilon}^\varepsilon\}.$$

Let $P = \{t_0, \dots, t_n\}$ partition s.t. $|P| < \delta$.

for δ to be chosen later.

Let $M = \sup_{[a,b]} |f(t)|$. If we let $P^* = P \cup P^\varepsilon$, then

$$\begin{aligned} U(P, f) - L(P, f) &= U(P, f) - U(P^*, f) \\ &\quad + U(P^*, f) - L(P^*, f) \\ &\quad + L(P^*, f) - L(P, f). \end{aligned}$$

P^* refinement of P^ε , So $(**)$ \Rightarrow

$$U(P^*, f) - L(P^*, f) < \frac{\varepsilon}{3}.$$

Claim: $\exists \delta$ s.t.

$$U(P, f) - U(P^*, f) < \varepsilon/3$$

$$L(P^*, f) - L(P, f) < \varepsilon/3.$$

Assuming, the claim, we are done! (11)

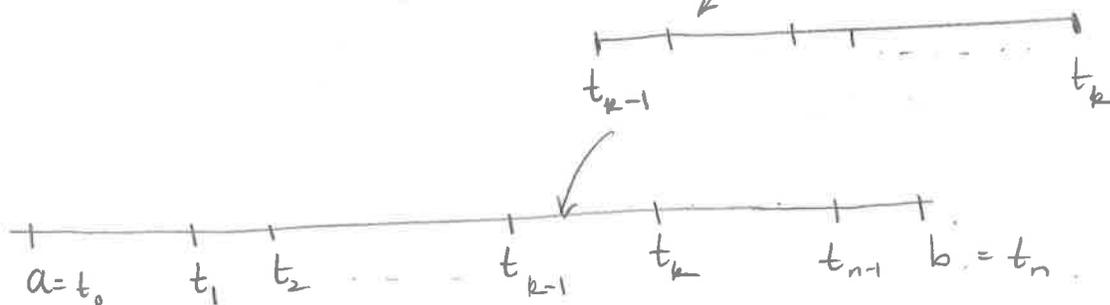
Since then $\exists \delta$ (from the claim)

s.t. $|P| < \delta \Rightarrow$

$$U(P, f) - L(P, f) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Pf of Claim: We prove the 2nd ineq. 1st is similar.

P.



let P_k^* be the partition of $[t_{k-1}, t_k]$ by points in P^* (and hence points in P^ϵ).

Then

$$L(P^*, f) = \sum_{k=1}^n L(P_k^*, f).$$

Moreover,
$$L(P, f) = \sum_{k=1}^n m_k \Delta t_k, \quad m_k = \inf_{[t_{k-1}, t_k]} f(t).$$

$$\Rightarrow L(P^*, f) - L(P, f) = \sum_{k=1}^n [L(P_k^*, f) - m_k \Delta t_k]$$

Now P^ϵ has at most n^ϵ partition points ⁽¹²⁾ different from P .

This means that $P_k^* = \{t_{k-1}, t_k\}$ for all k but at most $n^{(\epsilon)}$ intervals $[t_{k-1}, t_k]$.

i.e

$$L(P_k^*, f) - m_k \Delta t_k = 0$$

except possibly $n^{(\epsilon)}$ terms. But we also have

$$L(P_k^*, f) - m_k \Delta t_k \leq (M - m_k) \Delta t_k.$$

$$< M \cdot \delta \quad (\text{since } |P| < \delta).$$

$$\Rightarrow L(P^*, f) - L(P, f) < M \cdot \delta \cdot n^\epsilon.$$

Choose $\delta = \epsilon / 3 \cdot M \cdot n^\epsilon$.

Then $|P| < \delta \Rightarrow L(P^*, f) - L(P, f) < \epsilon / 3$.

proving the claim, and hence (2) \Rightarrow (3)

(3) \Rightarrow (1). For any partition, P .

$$L(P, f) \leq \int_a^b f(t) dt \leq \int_a^b f(t) dt \leq U(P, f).$$

(***)

Given $\epsilon > 0$, (3) $\Rightarrow \exists P$ s.t.

(13)

$$U(P, f) - L(P, f) < \epsilon.$$

Simply take

$$P = \{a = t_0 < t_1 = a + \frac{b-a}{N} \dots < t_n = b\}$$

where $\frac{b-a}{N} < \delta$.

But then

$$(***) \Rightarrow 0 \leq \int_a^b f(t) dt - \int_a^b f(t) dt < \epsilon.$$

Since this is true $\forall \epsilon > 0$

$$\int_a^b f(t) dt = \int_a^b f(t) dt$$

and $f \in R[a, b]$.

• Some sufficient conditions for integrability

Th^m 5.3: Let $f: [a, b] \rightarrow \mathbb{R}$ monotonic & bounded.

Then $f \in R[a, b]$.

Pf: Let $P = \{t_0 = a < t_1 = a + \frac{b-a}{n} < \dots < t_n = b\}$

where n is chosen big enough.

Sps $f \uparrow$ Then $M_k = \sup_{[t_{k-1}, t_k]} f(t) = f(t_k)$

$m_k = \inf_{[t_{k-1}, t_k]} f(t) = f(t_{k-1})$.

Also $\Delta t_k = \frac{1}{n}$.

(14)

$$\begin{aligned} \Rightarrow U(P, f) - L(P, f) &= \frac{1}{n} \sum_{k=1}^n f(t_k) - f(t_{k-1}) \\ &= \frac{1}{n} [f(t_1) - f(t_0) + f(t_2) - f(t_1) + \dots \\ &\quad \dots + f(t_{n-1}) - f(t_{n-2}) + f(t_n) - f(t_{n-1})] \\ &= \frac{1}{n} [f(t_n) - f(t_0)] = \frac{f(b) - f(a)}{n} < \varepsilon \end{aligned}$$

if n is chosen big enough

Th^m 5.4 Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded. Sps. the set of discont. of f are ^{at most} countable.

Then $f \in R[a, b]$.

Pf: Let $D \subset [a, b]$ be the set of discont. Here we prove it in the case that D is finite. Then the general case is proved in the appendix.

So suppose $D = \{d_1, \dots, d_n\}$. Let $\varepsilon > 0$ and $\eta > 0$ which we will choose later to depend on ε . Let $M = \sup |f(x)|$.

Let (a_j, b_j) be disjoint intervals around

$$\text{dn s.t. } \boxed{\sum (b_j - a_j) < \eta} \quad (*)$$

Let $B = \bigcup_{n=1}^N (a_j, b_j)$ be the "bad" set.

and let $G = [a, b] \setminus B$ be the "good" set.

G is compact and f is cont on G .

$\Rightarrow f$ is u.c. cont on G .

$\Rightarrow \exists \delta$ s.t. $\forall s, t \in G$,

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| < \eta. \quad (**)$$

Now, consider partition $P = \{t_0, t_1, \dots, t_n\}$ s.t.

(1) $a_j, b_j \in P$.

(2) No point in (a, b) belongs to P .

(3) If $t_{k-1} \neq a_j$ (and so $t_k \neq b_j$) then

$$\Delta t_k < \delta.$$

So essentially "slice up" G into intervals of size $< \delta$ & add points a_j, b_j to the partition. Note $\forall k$

$$M_k - m_k \leq 2M.$$

$$\forall k \text{ s.t. } t_{k-1} \in G, \quad M_k - m_k < \eta.$$

$$\begin{aligned} \Rightarrow U(P, f) - L(P, f) &= \sum_{k=1}^n (M_k - m_k) \Delta t_k \quad (16) \\ &= \sum_{t_{k-1} \neq a_j} \underbrace{(M_k - m_k)}_{\leq \eta} \Delta t_k + \sum_{t_{k-1} = a_j} \overbrace{(M_k - m_k)}^{\leq 2M} (b_j - a_j) \\ &\leq \eta \cdot \sum_{t_{k-1} \neq a_j} \Delta t_k + 2M \cdot \sum (b_j - a_j) \\ &\leq \eta(b-a) + 2M \cdot \eta = \eta(b-a+2M). \end{aligned}$$

If we let $\eta < \frac{\varepsilon}{b-a+2M}$, then

$$U(P, f) - L(P, f) < \varepsilon.$$

Properties of the integral.

Thm 5.5: 1) If $f_1, f_2 \in \mathcal{R}[a, b]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in \mathcal{R}[a, b]$ and

$$\int_a^b c_1 f_1 + c_2 f_2 = c_1 \int_a^b f_1 + c_2 \int_a^b f_2$$

2) $f_1 \leq f_2$ on $[a, b]$ & $f_1, f_2 \in \mathcal{R}[a, b] \Rightarrow$

$$\int_a^b f_1 \leq \int_a^b f_2$$

3) $f \in R[a, b]$ and $a < c < b$, then $f \in R[a, c]$ and $f \in R[c, b]$ and

$$\int_a^b f(t) dt = \int_a^c f(t) dt + \int_c^b f(t) dt.$$

Pf is left as an exercise

Th^m 5.6: Sp^s $f \in R[a, b]$, $m \leq f \leq M$ and $\varphi: [m, M] \rightarrow \mathbb{R}$ is continuous. Then $\varphi \circ f \in R[a, b]$.

Rk: If φ is merely assumed to be integrable

Consider $f = \begin{cases} 0, & x \notin \mathbb{Q} \\ y/n, & x = m/n, (m, n) = 1. \end{cases}$ $\varphi(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$

Then $\varphi \circ f = \begin{cases} 1, & x \notin \mathbb{Q} \\ 0, & x \in \mathbb{Q} \end{cases} \notin R[0, 1].$

Cor: (1) $f, g \in R[a, b] \Rightarrow f \cdot g \in R[a, b]$

(2) $f \in R[a, b] \Rightarrow |f| \in R[a, b]$. Moreover

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f| dt$$

Pf: (1) Apply to $\varphi(x) = x^2$. Then $f^2, g^2, (f+g)^2 \in R$.

$$\Rightarrow f \cdot g = \frac{1}{2} [(f+g)^2 - f^2 - g^2] \in R[a, b].$$

(2) Apply theorem to $\varphi(t) = |t|$. So
 $f \in R[a, b] \Rightarrow |f| \in R[a, b]$.

Let $c = \pm 1$ s.t. $c \int_a^b f(t) dt \geq 0$.

Then
$$\left| \int_a^b f(t) dt \right| = c \int_a^b f(t) dt = \int_a^b c \cdot f dt \leq \int_a^b |f| dt \text{ since } cf \leq |f|.$$

Pf of Th^m 5.7: Let $\epsilon > 0$ and $\eta > 0$ (to be chosen later).

Let $h(t) = \varphi \circ f(t)$. Since φ is u.c on $[m, M]$, $\exists \delta < \eta$ s.t.

$$|x - y| < \delta \Rightarrow |\varphi(x) - \varphi(y)| < \eta \quad (*)$$

$f \in R[a, b] \Rightarrow \exists P = \{t_0, \dots, t_n\}$ s.t.

$$U(P, f) - L(P, f) < \delta^2 \quad (**).$$

Let $K = \sup |\varphi(t)|$.

Claim: If $\eta = \epsilon / (b - a + 2K)$, then

$$U(P, h) - L(P, h) < \epsilon.$$

P.f. Let $M_k = \sup_{[t_{k-1}, t_k]} f$, $m_k = \inf_{[t_{k-1}, t_k]} f$

$$M_k^* = \sup_{[t_{k-1}, t_k]} h, \quad m_k^* = \inf_{[t_{k-1}, t_k]} h.$$

Good Set / Bad Set

$$G = \{k \in \{1, \dots, n\} \mid M_k - m_k < \delta\}$$

$$B = \{1, \dots, n\} \setminus G$$

Recall

$$U(P, h) - L(P, h) = \sum_k (M_k^* - m_k^*) \Delta t_k$$

CASE 1: $k \in G$.

Here $M_k - m_k < \delta \Rightarrow |f(s) - f(t)| < \delta \quad \forall s, t \in [t_{k-1}, t_k]$

$$(*) \Rightarrow |h(s) - h(t)| < \eta \quad \forall s, t \in [t_{k-1}, t_k].$$

So
$$\sum_{k \in G} (M_k^* - m_k^*) \Delta t_k < \eta \cdot (b - a).$$

CASE 2: $k \in B$.

Here $M_k^* - m_k^* \leq 2K$ since $|h| \leq K$.

Also, for $k \in B$, $M_k - m_k \geq \delta$, So

$$\delta \sum_{k \in B} \Delta t_k \leq \sum_{k \in B} (M_k - m_k) \Delta t_k$$

$$\leq \sum_{k=1}^n (M_k - m_k) \Delta t_k$$

$$= U(P, f) - L(P, f) < \delta^2$$

$$\Rightarrow \sum_{k \in B} \Delta t_k < \delta$$

$$\Rightarrow \sum_{k \in B} (M_k^* - m_k^*) \Delta t_k < 2K \cdot \delta < 2K \cdot \eta$$

Together $U(P, h) - L(P, h) < \eta [b - a + 2K] = \epsilon$.

So claim, and hence theorem is proved.

Integration and differentiation

Th^m 5.8 (1st fundamental Th^m): If $f \in R[a, b]$, consider $F: [a, b] \rightarrow \mathbb{R}$ defined by.

$$F(x) := \int_a^x f(t) dt$$

Then:

(a) F is cont. on $[a, b]$.

(b) If f is cont. at $p \implies F$ is diff at p and

$$F'(p) = f(p).$$

Pf (1). Sps $|f(t)| \leq M$ on $[a, b]$. Then ⁽²¹⁾ ~~(49)~~ if $x \leq y$.

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f(t) dt - \int_a^x f(t) dt \right| \\ &= \left| \int_x^y f(t) dt \right| \\ &\leq \int_x^y |f(t)| dt \leq M(y-x). \end{aligned}$$

More generally for any $x, y \in [a, b]$

$$|F(y) - F(x)| \leq M|y-x|.$$

Given $\varepsilon > 0$, let $\delta < M/\varepsilon$. Then

$$|x-y| < \delta \Rightarrow |F(y) - F(x)| < \varepsilon.$$

So F is u.c on $[a, b]$.

(2) Let $\varphi(x)$ be the diff. quotient of F at p . Sps $x > p$.

$$\varphi(x) = \frac{F(x) - F(p)}{x-p}$$

$$= \frac{1}{(x-p)} \int_p^x f(t) dt.$$

$$\int_p^x f(t) dt = - \int_x^p f(t) dt \quad \text{if } x < p.$$

Also
$$f(p) = \lim_{x \rightarrow p} \frac{1}{(x-p)} \int_p^x f(t) dt.$$

$$\Rightarrow |Q(x) - f(p)| = \frac{1}{|x-p|} \left| \int_p^x [f(t) - f(p)] dt \right| \quad (22)$$

Given $\varepsilon > 0$, $\exists \delta$ s.t.

$$|t-p| < \delta \Rightarrow |f(t) - f(p)| < \varepsilon$$

Sps $|x-p| < \delta$, then since t between p & x

$$|t-p| < \delta \quad \text{So}$$

$$|x-p| < \delta \Rightarrow |Q(x) - f(p)| = \frac{1}{|x-p|} \left| \int_p^x f(t) - f(p) \right|$$

$$\leq \frac{1}{|x-p|} \int_p^x |f(t) - f(p)| dt$$

$$\leq \frac{\varepsilon}{|x-p|} (x-p) = \varepsilon \quad \text{since } x > p$$

$$\Rightarrow \lim_{x \rightarrow p^+} Q(x) = f(p)$$

Similarly $\lim_{x \rightarrow p^-} Q(x) = f(p)$

$$\Rightarrow \lim_{x \rightarrow p} Q(x) = f(p) \Rightarrow F'(p) = f(p)$$

Defⁿ: A diff function $F: [a, b] \rightarrow \mathbb{R}$ s.t. (23)
 $F'(x) = f(x)$ is called an indefinite
integral of f or anti derivative of f . $F = \int f$

Cor 5.9: (2nd fundamental theorem). If f is
cont. and F an anti-derivative, then

$$\int_a^b f(t) dt = F(b) - F(a) := F(t) \Big|_a^b$$

Pf: let $G(x) = \int_a^x f(t) dt$, $G(a) = 0$
 $G(b) = \int_a^b f(t) dt$

Then Th^m $\Rightarrow G$ is diff on $[a, b]$

$$\& G'(x) = f(x)$$

$$\Rightarrow G(x) - F(x) = \text{const.}$$

$$\Rightarrow G(b) = F(b) - F(a).$$

Done!

• Examples: 1) Recall

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0.$$

$$\Rightarrow \int x^\alpha dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1}, & \alpha \neq -1 \\ \ln|x|, & \alpha = -1 \end{cases}$$

(24)

$$2) \int \sin(x) = -\cos x + C, \quad \int \cos x = \sin x + C$$

Thm 5.10. (Change of var.) Sps $u \in C^1[a, b]$, $u'(t) \neq 0 \forall t$ and $u[a, b] \subset I$, I an interval s.t.

$f: I \rightarrow \mathbb{R}$ is cont. Then

$$\int_{u(a)}^{u(b)} f(u) du = \int_a^b f(u(t)) \cdot u'(t) dt.$$

Pf: Sps $u' > 0$. Then consider, $x \in [u(a), u(b)]$

$$F(x) = \int_{u(a)}^x f(u) du.$$

Then $F'(u) = f(u)$. By Chain rule.

$$\begin{aligned} \frac{d}{dt} F(u(t)) &= F'(u(t)) \cdot u'(t) \\ &= f(u(t)) \cdot u'(t). \end{aligned}$$

Integrating

$$F(u(b)) - F(u(a)) = \int_a^b f(u(t)) \cdot u'(t) dt.$$

$$\Rightarrow \int_{u(a)}^{u(b)} f(u) du = \int_a^b f(u(t)) \cdot u'(t) dt.$$

Rk: At the level of indefinite integrals

$$\int f(u) du = \int f(u(t)) \cdot u'(t) dt.$$

Example: $\int \tan(t) dt = \int \frac{\sin(t)}{\cos(t)} dt$

Let $u = \cos(t)$. Then $du = u'(t) dt = -\sin(t) dt$.

So $\int \tan t dt = - \int \frac{du}{u} = - \ln|u|$

$$= \ln|\sec t|.$$

Th^m 5.11 (Integration by parts): If $f, g \in C^1[a, b]$.

Then $\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b g \cdot f'$

Pf: $\frac{d}{dt} f \cdot g = f' \cdot g + g' \cdot f$

Integrating $\int_a^b (f \cdot g)' = \int_a^b f' \cdot g + \int_a^b g' \cdot f$

$$(f \cdot g)(b) - (f \cdot g)(a)$$

Rk: Easier way to remember,

$$u = f' \cdot dt, \quad v = g' \cdot dt$$

$$u = f, \quad v = g.$$

Then

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Example: $\int \ln(t) dt$, $u = \ln(t)$, $dv = dt$
 $du = \frac{dt}{t}$, $v = t$

$$\text{So } \int \ln(t) dt = t \cdot \ln t - \int t \cdot \frac{dt}{t}$$

$$= t \ln t - t$$

• Improper integrals This is to define integrals on unbounded domains or for unbounded functions.

Defⁿ: Let $-\infty \leq a \leq b \leq \infty$. We say f is locally integrable if $\forall a < c < d < b$, $f \in R[c, d]$. We say f is integrable on (a, b) if it is locally integrable, and

$$\lim_{c \rightarrow a^+} \left[\lim_{d \rightarrow b^-} \int_c^d f(t) dt \right]$$

exists and is finite. We also say $\int_a^b f$ is convergent.

Rk: It turns out that if f is integrable on (a, b) . Then (27)

$$\lim_{c \rightarrow a^+} \lim_{d \rightarrow b^-} \int_c^d f(t) dt = \lim_{d \rightarrow b^-} \lim_{c \rightarrow a^+} \int_c^d f(t) dt.$$

Examples: 1) $\int_a^\infty x^p dx$.

Clearly $x^p \in R[1, c] \forall c \in \mathbb{R}$. Moreover, if $p \neq -1$.

$$\int_a^c x^p dx = \left. \frac{x^{p+1}}{p+1} \right|_a^c = \frac{c^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}.$$

This has a finite limit as $c \rightarrow \infty \iff$

$$p+1 < 0 \implies p < -1.$$

if $p = 1$, then

$$\int_a^c \frac{dx}{x} = \frac{\ln c}{- \ln a} \xrightarrow{c \rightarrow \infty} \infty$$

$$\text{So } \int_a^\infty x^p = \begin{cases} -\frac{a^{p+1}}{1+p}, & p < -1. \\ \text{DNE}, & p \geq -1. \end{cases}$$

2) $\int_0^a x^p dx$. Again clearly $x^p \in R[c, a]$

$$\forall a > c > 0.$$

Moreover

$$\int_c^a x^p dx = \begin{cases} \frac{a^{p+1}}{1+p} - \frac{c^{p+1}}{p+1} & , p \neq -1 \\ -\ln c & , p = -1. \end{cases}$$

Again limit as $c \rightarrow 0^+$ exists iff $p+1 > 0$
 or $p > -1$.

So

$$\int_c^1 x^p dx = \begin{cases} \frac{1}{1+p} & , p > -1. \\ \text{DNE} & p < -1. \end{cases}$$

Th^m 5-12: (Comparison principle) If $f(x) \geq g(x) \geq 0$
 $\forall x \in (a, b)$, and f, g locally integrable on (a, b) .

- 1) $\int_a^b f$ convergent $\implies \int_a^b g$ convergent.
- 2) $\int_a^b g$ divergent $\implies \int_a^b f$ divergent.