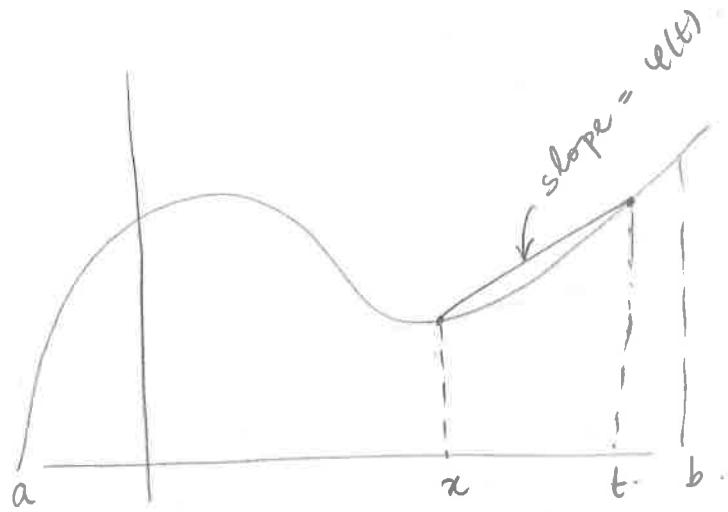


## 4. Differentiation

• The derivative: Let  $f: [a, b] \rightarrow \mathbb{R}$



Define  $\varphi(t) := \varphi_{f, x}(t) := \frac{f(t) - f(x)}{t - x}$  for  $t \neq x$

$\varphi(t)$  is called difference quotient.

Def<sup>n</sup>: For  $x \in [a, b]$  we say  $f$  is differentiable

at " $x$ " if

$$f'(x) := \lim_{t \rightarrow x} \varphi(t)$$

exists. We then call  $f'$  the derivative of  $f$  at  $x$ . If  $x = a$  or  $b$ , we interpret the limit as a one-sided limit.

Rk: Geometrically,  $f'(x)$  is the slope of the tangent line to  $y = f(x)$  at  $(x, f(x))$ .

Notation:  $f'(x) = \frac{df}{dx} = \left. \frac{d}{dt} f(t) \right|_{t=x}$

Examples: 1) Constants:  $f(t) = c \quad \forall t \in [a, b]$

Then  $f(t) = 0$  for any  $x$ . So

$$f'(x) \equiv 0 \text{ on } [a, b].$$

2) Monomials:  $P_n(t) = t^n, n \in \mathbb{N}$ ,

If  $n=0$ ,  $P_0(t)=1$ , so  $P_0'(x)=0$ .

Sps  $n \geq 1$ . Then for  $t \neq x$

$$\varphi(t) = \frac{P_n(t) - P_n(x)}{t - x} = \frac{t^n - x^n}{t - x}$$

$$= t^{n-1} + t^{n-2}x + \dots + x^{n-1}$$

$$\text{So } P_n'(x) = \lim_{t \rightarrow x} \varphi(t) = n \cdot x^{n-1}.$$

$$\Rightarrow \boxed{P_n'(x) = n \cdot x^{n-1}}$$

Note that the above formula also holds for

$$n=0.$$

If  $\frac{d}{dx} t^n = n t^{n-1}$

It follows from the generalized binomial theorem, and an argument as above, that

Thm 4.1 For any  $\alpha \in \mathbb{R}$ ,

$$\boxed{\frac{d}{dx} x^\alpha = \alpha x^{\alpha-1}}$$

Later, we will prove this using chain rule.

3) Trig functions:  $\frac{d}{dx} \sin x = \cos x$

$$\frac{d}{dx} \cos x = -\sin x$$

Later we will define  $\sin(x)$  &  $\cos(x)$  using power series, and we will prove these formulae.

4) Exp. Later we will define  $e^x$  as a power series, and will show

$$\frac{d}{dx} e^x = e^x$$

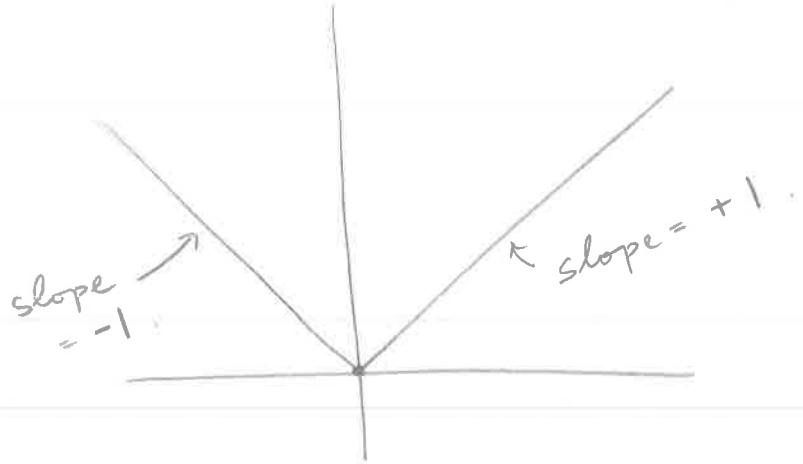
5) Absolute value:  $f(x) = |x|$  on  $\mathbb{R}$

$$= \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

(Clearly  $f(x)$  is diff.  $\forall x \neq 0$ . At  $x=0$ )

if  $t > 0$ ,  $\varphi(t) = \frac{f(t) - f(0)}{t} = 1$

$t < 0$   $\varphi(t) = \frac{f(t) - f(0)}{t} = -1$



So  $\lim_{t \rightarrow 0} \varphi(t)$  DNE

$\Rightarrow f(x)$  is NOT diff at  $x = 0$

Rk: This is prototypical of non differential functions, that the graphs have corners.

#### • Some elementary properties

Th<sup>m</sup> 4.2: If  $f: [a, b] \rightarrow \mathbb{R}$  is diff at  $x \in [a, b]$   
 $\Rightarrow f$  is cont. at  $x$ .

Pf: Note that for  $t \neq x$ .

$$f(t) - f(x) = \varphi(t)(t-x) \xrightarrow{t \rightarrow x} 0$$

So  $\lim_{t \rightarrow x} f(t) = f(x)$

and  $f$  is cont. at  $x$ .

Thm 4.3 Sps  $f, g: [a, b] \rightarrow \mathbb{R}$ , and diff at  $x \in [a, b]$ . Then  $f \pm g, f \cdot g$  are diff at  $x$ .  $f/g$  is also diff at  $x$  if  $g(x) \neq 0$ . Moreover -

$$(a) (f \pm g)'(x) = f'(x) \pm g'(x).$$

$$(b) (f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

$$(c) \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2}.$$

Pf: (a) follows from analogous statement on limits.

(b) Let  $\varphi, \psi$  be diff quotients of  $f$  and  $g$ .  
Let  $h(t) = f(t)g(t)$ . Then

$$\begin{aligned} h(t) - h(x) &= f(t)g(t) - f(x)g(x) \\ &= f(t) \cdot [g(t) - g(x)] + g(t) [f(t) - f(x)] \end{aligned}$$

$$\Rightarrow \frac{h(t) - h(x)}{t - x} = f(t) \cdot \varphi(t) + g(t) \psi(t) \xrightarrow{t \rightarrow x} f(x)g'(x) + g(x)f'(x).$$

(c) Now let  $p(t) = f(t)/g(t)$ . If  $g(x) \neq 0$ , since  $g$  is cont.,  $g(t) \neq 0$  for  $t$  close to  $x$ .

Then for  $t \neq x$ .

$$p(t) - p(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(t)g(x) - f(x)g(t)}{g(t) \cdot g(x)}$$

$$= \frac{f(t)g(x) - f(t)g(t) + f(t)g(t) - f(x)g(t)}{g(t) \cdot g(x)}$$

$$= \frac{f(t)[g(x) - g(t)] + g(t)[f(t) - f(x)]}{g(t) \cdot g(x)}$$

$$\Rightarrow \frac{p(t) - p(x)}{t - x} = \frac{g(t) \cdot \psi(t) - f(t) \psi(t)}{g(t) \cdot g(x)}$$

$$\xrightarrow{t \rightarrow x} \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Thm 4.4 (Chain rule). Suppose  $f: [a, b] \rightarrow \mathbb{R}$ ,  $f'(x)$  exists for some  $x \in [a, b]$ ,  $g$  is defined on some interval  $I$  s.t  $f([a, b]) \subset I$ , and  $g$  is diff. at  $f(x)$ . Then

$$h(t) = g \circ f(t)$$

is diff at  $x$ .

"Naive proof":  $\frac{h(t) - h(x)}{t - x} = \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \cdot \frac{f(t) - f(x)}{t - x}$

Problem:  $f(t)$  might be equal to  $f(x)$  even if  $t \neq x$ .

Pf: let  $y = f(x)$ . Then  $f'(x)$  &  $g'(y)$  exist. So

$$f(t) - f(x) = (t - x)[f'(x) + r(t)] \quad \forall t \in [a, b]$$

$$g(s) - g(y) = (s - y)[g'(y) + \sigma(s)] \quad \forall s \in I$$

$$\text{where } \lim_{t \rightarrow x} r(t) = \lim_{s \rightarrow y} \sigma(s) = 0.$$

let  $s = f(t)$ . Then

$$\begin{aligned} h(t) - h(x) &= g(s) - g(y) \\ &= (s - y)[g'(y) + \sigma(f(t))] \\ &= (t - x)[f'(x) + r(t)][g'(y) + \sigma(f(t))] \end{aligned}$$

So for  $t \neq x$

$$\frac{h(t) - h(x)}{t - x} = [f'(x) + r(t)][g'(y) + \sigma(f(t))]$$

At  $t \rightarrow x$ ,  $s \rightarrow y$  since  $f$  is cont. at  $x$

$$\Rightarrow \lim_{t \rightarrow x} \frac{h(t) - h(x)}{t - x} = f'(x) \cdot g'(y)$$

Examples 1).  $h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & x = 0 \end{cases}$

For  $x \neq 0$ , by chain rule, quotient rule & product rule

$$\begin{aligned} h'(x) &= 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \\ &= 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right). \quad (x \neq 0) \end{aligned}$$

At  $x = 0$

$$\lim_{t \rightarrow 0} \frac{h(t) - h(0)}{t} = \lim_{t \rightarrow 0} \frac{t^2 \sin\left(\frac{1}{t}\right)}{t} = \lim_{t \rightarrow 0} t \sin\left(\frac{1}{t}\right) = 0$$

$$\text{So } h'(0) = 0$$

But  $\lim_{x \rightarrow 0} h'(x)$  DNE!

2) logarithmic diff:  $y = \ln(x)$  is defined to be the unique number s.t  $e^y = x$ .

We'll see later that

$$L(x): (0, \infty) \rightarrow \mathbb{R} \\ x \mapsto \ln x.$$

is diff  $\forall x \in (0, \infty)$ .

Aim Compute  $L'(x)$ .

We know

$$e^{L(x)} = x \\ \Rightarrow \frac{d}{dx} e^{L(x)} = 1$$

$$\text{Chain rule } \Rightarrow \frac{d}{dx} e^{L(x)} = \frac{d}{dy} e^y \cdot \frac{dL(x)}{dx}, \quad y = L(x) \\ = e^{L(x)} \cdot L'(x) \\ = x \cdot L'(x)$$

So

$$\boxed{L'(x) = \frac{1}{x}}$$

Pf of Th<sup>m</sup>4.1: Let  $y = x^\alpha$   
 $\Rightarrow \ln y = \alpha \cdot \ln x.$

Diff w.r.t 'x',  $\frac{y'}{y} = \frac{\alpha}{x}$ .

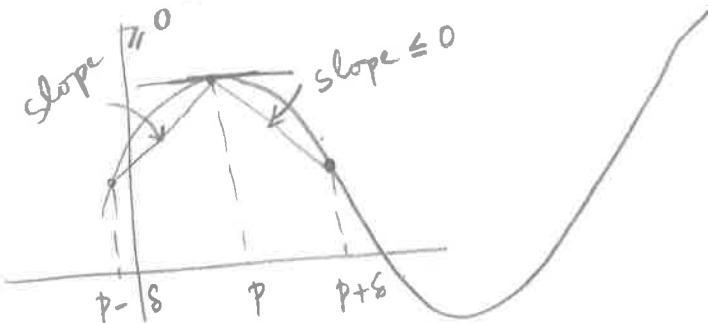
$$\Rightarrow y' = \frac{\alpha y}{x} = \alpha \cdot x^{\alpha-1}.$$

### • Mean Value Theorem

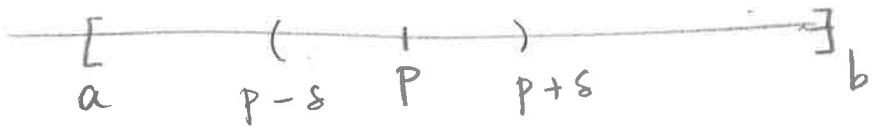
Def<sup>n</sup>: (Local Max/Min). Let  $(X, d)$  be a metric sp., and  $f: X \rightarrow \mathbb{R}$ . We say  $f$  has a local max (resp. local min) at  $p \in X$  if  $\exists s > 0$  s.t.  $\forall x \in B_s(p)$

$$f(p) \geq f(x) \quad (\text{resp. } f(p) \leq f(x)).$$

Th<sup>m</sup>4.5. Let  $f: [a, b] \rightarrow \mathbb{R}$ . If  $f$  has a local max or local min at  $p \in (a, b)$ . If  $f$  is diff at  $p \Rightarrow f'(p) = 0$



Pf: Sps  $p$  is a local max. Similar arg.  
also works for local min. Then  $\exists s$  s.t  
 $f(p) \geq f(t) \forall t \in (p-s, p+s)$ .



$$\text{If } t \in (p-s, p) \Rightarrow \varphi(t) = \frac{f(t) - f(p)}{t - p} \geq 0$$

$$t \in (p, p+s) \Rightarrow \varphi(t) = \frac{f(t) - f(p)}{t - p} \leq 0$$

$$\text{At } t \rightarrow p \quad \varphi(p+) \leq 0$$

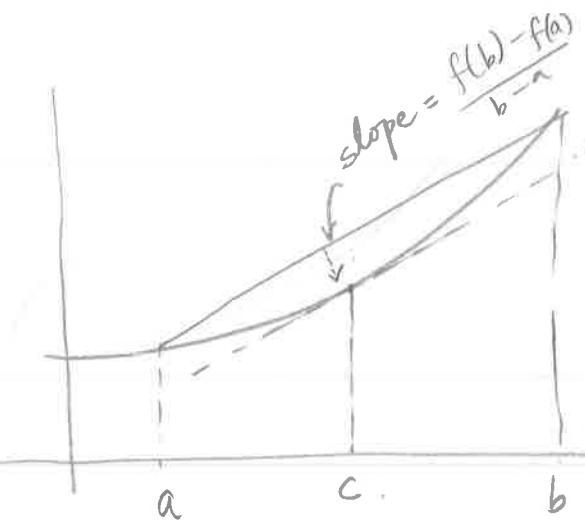
$$\varphi(p-) \geq 0$$

$$f \text{ diff at } p \Rightarrow \varphi(p+) = \varphi(p-) = f'(p)$$

$$\text{So } f'(p) = 0.$$

Th<sup>m</sup> 4.6 (Mean value theorem, MVT) If  $f: [a, b] \rightarrow \mathbb{R}$  is cont.  
and  $f$  is diff. on  $(a, b)$ . Then  $\exists c \in (a, b)$  s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Pf: Let  $h: [a, b] \rightarrow \mathbb{R}$ , defined by

$$h(t) = [f(b) - f(a)]t - (b-a)f(t).$$

Then ①  $h$  is cont. on  $[a, b]$  & diff. on  $(a, b)$ .

$$\textcircled{2} \quad h(a) = h(b)$$

$$\textcircled{3} \quad h'(t) = f(b) - f(a) - (b-a)f'(t).$$

Aim: Show that  $\exists c \in (a, b)$  s.t.  $h'(c) = 0$

CASE 1:  $h$  is const.

Then  $h'(t) = 0 \neq t$ , so nothing to prove.

CASE 2:  $h$  is non-constant

Extremum value theorem  $\Rightarrow \exists$  max & min of  $h$  in  $[a, b]$ , say  $p$  and  $q$  resp. Since  $h(a) = h(b)$ , one of these points lies in  $(a, b)$ ; say,  $c$ . Then by Thm 4.5,  $h'(c) = 0$ , and we are done!

Rk:  $f$  has to be cont on  $[a, b]$ . Else, consider

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 1, & x = 0 \end{cases}$$

Then  $f'(x) = 1 \nabla x \in (0, 1)$  but  $\frac{f(1) - f(0)}{1 - 0} = 0$

Cor 4.7: Let  $f$  be cont. on  $[a, b]$  and diff on  $(a, b)$ .

Then (1)  $f$  is const  $\Leftrightarrow f'(x) = 0 \nabla x \in (a, b)$ .

(2)  $f \uparrow \Leftrightarrow f'(x) \geq 0 \nabla x \in (a, b)$

(3)  $f \downarrow \Leftrightarrow f'(x) \leq 0 \nabla x \in (a, b)$

Pf: (1)  $\Rightarrow$  Trivial

$\Leftarrow$  If not, then  $\exists p, q \in [a, b], p < q$  &  $f(p) \neq f(q)$ . Apply Thm 4.6 to  $f: [p, q] \rightarrow \mathbb{R}$ .

Then  $\exists c \in (p, q)$  s.t.

$$0 = f'(c) = \frac{f(q) - f(p)}{q - p} \neq 0$$

Contradiction!

(2)  $\Rightarrow$  Sps  $f \uparrow$ . Recall  $\varphi(t) = \frac{f(t) - f(x)}{t - x}$

If  $t > x$ ,  $f(t) \geq f(x)$  since  $f \uparrow$

and so  $\varphi(t) \geq 0$ .

Similarly if  $t < x$ ,  $\varphi(t) \geq 0$

$$\Rightarrow f'(x) = \lim_{t \rightarrow x} \varphi(t) \geq 0$$

$\Leftarrow$  Sps  $f'(x) \geq 0$  on  $[a, b]$ . For any  $s, t \in (a, b)$  with  $s < t$ ,  $\exists c \in (s, t)$  s.t

$$0 \leq f'(c) := \frac{f(t) - f(s)}{t - s}.$$

$$\Rightarrow f(t) \geq f(s)$$

So  $f \uparrow$

(3) Similar to (2)

Cor 4.8 (Generalized mean value theorem, GMVT)

if  $f, g: [a, b] \rightarrow \mathbb{R}$  cont. and diff on  $(a, b)$ .

Then  $\exists c \in (a, b)$  s.t

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

Pf: Let  $h(t) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$

$$\begin{aligned} \text{Then } h(a) &= f(b)g(a) - g(b)f(a) \\ &= h(b). \end{aligned}$$

Thm 4.6  $\Rightarrow \exists c$  s.t  $h'(c) = 0$

Done!

Cor 4.9 ( $L'$ Hopital's rule). Sps  $a < c < b$ , and  $f$  and  $g$  are diff functions on  $(a, b) / \{c\}$ . s.t

$g'(x) \neq 0 \quad \forall x$ . If

$$(1) \quad \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x) = 0$$

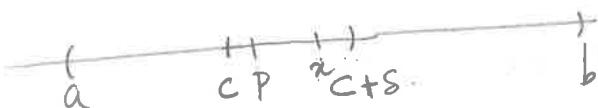
$$(2) \quad \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} = L$$

Then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = L$$

Pf: Claim 1:  $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = L$

Pf: Let  $\varepsilon > 0$   $\exists (2) \Rightarrow \exists s$  s.t.



$$\left| \frac{f'(t)}{g'(t)} - L \right| < \varepsilon \quad \forall t \in (c, c+s)$$

Let  $x \in (c, c+s)$  and  $c < p < x < c+s$ .

Cor 4.8  $\Rightarrow \exists t \in (p, x)$  s.t

$$\frac{f'(t)}{g'(t)} = \frac{f(x) - f(p)}{g(x) - g(p)}$$

$$\Rightarrow \left| \frac{f(x) - f(p)}{g(x) - g(p)} - L \right| < \varepsilon$$

Letting  $p \rightarrow c^+$ ,  $\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$ .

So  $\forall x \in (c, c+\delta)$ ,  $\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$ .

or  $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = L$ .

Claim 2:  $\lim_{x \rightarrow c^-} \frac{f(x)}{g(x)} = L$ .

Pf: Similar.

So  $\lim_{x \rightarrow c^-} \frac{f(x)}{g(x)} = L$ .

• Higher order derivatives & Taylor's theorem.

We denote higher order derivatives of  $f$  by  $f', f'', \dots, f^{(n)}$  etc. if they exist.

Note if  $f^{(k)}$  exists at ' $x$ '  $\Rightarrow f, f', \dots, f^{(k-1)}$  are cont at ' $x$ '.

Def<sup>n</sup>: We say  $f \in C^k[a, b]$  if  $f', \dots, f^{(k)}$  exist on  $[a, b]$ .

e.g. Polynomials.

Th<sup>m</sup> (Taylor's theorem). Sps  $f \in C^n[a, b]$ . For any  $x, p \in (a, b)$ ,  $\exists c$  between  $p$  &  $x$  s.t.

$$f(x) = T_{n-1}(x) + \frac{f^{(n)}(c)}{n!} (x - p)^n,$$

where

$$T_{n-1}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(p)}{k!} (x - p)^k$$

is the  $(n-1)^{st}$  Taylor polynomial. Consequently if  $R_{n-1}(x) = f(x) - T_{n-1}(x)$  is the remainder, we have the estimate

$$|R_{n-1}(x)| \leq \frac{M}{n!} |x - p|^n$$

where  $M = \sup_{[a,b]} |f^{(n)}(t)|$ .

Pf. Let  $M = [f(x) - T_{n-1}(x)] / (x - p)^n$ ,  $x \neq p$ , and

$$\text{let } g(y) = f(y) - T_{n-1}(y) - M(y - p)^n$$

Note:  $f^{(k)}(p) = T_{n-1}^{(k)}(p) \quad \forall k = 0, 1, \dots, n-1$ .

So  $g$  has the foll. properties:  $T_{n-1}^{(n)}(x) = 0 \quad \forall x$ .

$$\textcircled{1} \quad g(x) = 0$$

$$\textcircled{2} \quad g(p) = g'(p) = \dots = g^{(n-1)}(p) = 0$$

$g(x) = g(p) = 0 \Rightarrow \exists y_1$  between  $x$  and  $p$   
s.t.  $g'(y_1) = 0$ .

$g'(p) = g'(y_1) = 0 \Rightarrow \exists y_2$  bet.  $p$  &  $y_1$  s.t.

$g''(y_2) = 0$ . and so, on.

Eventually,  $\exists y_{n-1}$  s.t.  $g^{(n-1)}(y_{n-1}) = 0$ .

Again MVT  $\Rightarrow \exists c$  between  $p$  &  $y_{n-1}$

s.t.  $g^{(n)}(c) = 0$ .

But  $g^{(n)}(y) = f^{(n)}(y) - n!M$ .

So  $\exists c$  between  $y_{n-1}$  &  $p$ , and hence  
between  $x$  &  $p$  s.t.

$$f^{(n)}(c) = n! \frac{[f(x) - T_{n-1}(x)]}{(x - p)^n}.$$

$\Rightarrow f(x) = T_{n-1}(x) + \frac{f^{(n)}(c)}{n!} (x - p)^n$  for some  $c$ .

Rk:  $n=1$  is the usual MVT.

## Applications of derivatives

We said that if  $f$  is diff on  $(a, b)$  &  $p$  is a local max/min, then  $f'(p) = 0$ .

Def<sup>n</sup>: Let  $f: (a, b) \rightarrow \mathbb{R}$  be a function. Then  $p \in (a, b)$  is called a critical point if either

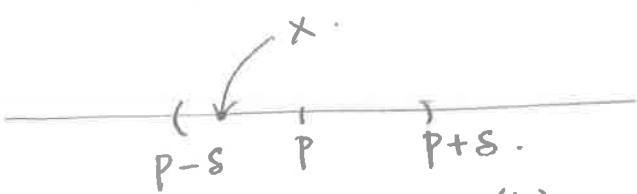
- $f$  is not diff at  $p$
- $f'(p) = 0$

Th<sup>m</sup>: Let  $f \in C^n[a, b]$ , and  $f'(p) = f''(p) = \dots = f^{(n-1)}(p) = 0$  but  $f^{(n)}(p) \neq 0$ . Sps  $n > 1$ . Then

- $n$  is even,  $f^{(n)}(p) > 0 \Rightarrow f$  has a local min at  $p$ .
- $n$  is even,  $f^{(n)}(p) < 0 \Rightarrow f$  has a local max at  $p$ .
- $n$  is odd.  $\Rightarrow f$  has neither a max or a min at  $p$  and is called an inflection point.

Pf: (a)  $f \in C^n[a, b]$  and  $f^{(n)}(p) > 0$ : Then continuity of  $f^{(n)}$  Why?  $\Rightarrow$  If  $s \leftarrow t \in (p-s, p+s)$

$$f^{(n)}(t) > 0.$$



Now, let  $x \in (p-s, p+s)$ . Since  $f^{(k)}(p) = 0$   $\forall k=1, \dots, n-1$ ,  $T_{n-1}(x) = f(p)$ . So Taylor's theorem  
 $\Rightarrow$   $\exists c$  between  $p$  &  $x$  & hence  $c \in (p-s, p+s)$

s.t

$$f(x) = f(p) + \frac{f^{(n)}(c)}{n!} \cdot (x-p)^n$$

$$c \in (p-s, p+s) \Rightarrow f^{(n)}(c) > 0.$$

$$n \boxed{\text{even}} \Rightarrow (x-p)^n > 0$$

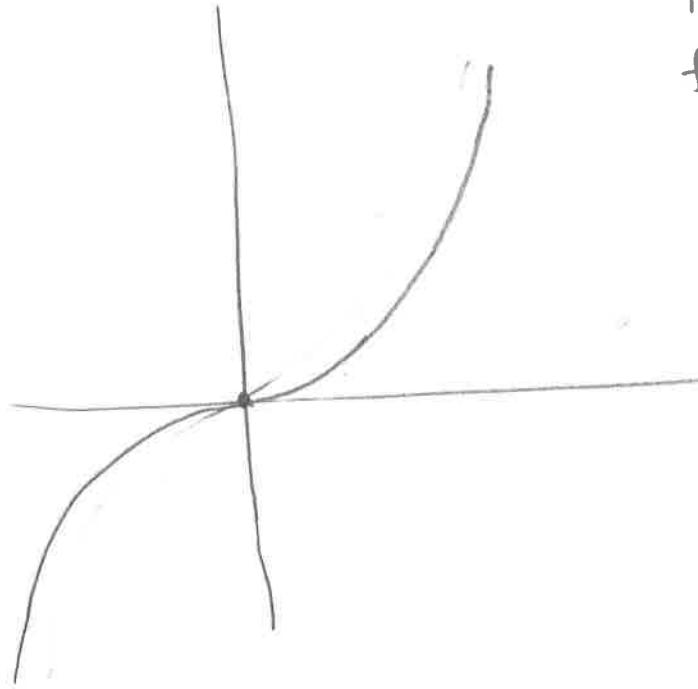
$$\Rightarrow f(x) > f(p)$$

True for any  $x \in (p-s, p+s) \Rightarrow p$  is local min

(b)  $L(c)$  are proved in similar way.

Prop: A cont  $f: [a, b] \rightarrow \mathbb{R}$  has an extremum  $\Leftrightarrow$   
at  $p \in [a, b]$ . Then either  $p=a$  or  $b$ , or  $p$  is a  
critical point. Furthermore if  $f''(p) > 0$  then  
 $p$  has a local min, and if  $f''(p) < 0$  then  
 $p$  has a local max at  $p$ .

Example: ①  $f(x) = x^3$ ,



$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

$$\text{So } f^{(3)}(0) \neq 0$$

$$f^{(k)}(0) = 0$$

$$\text{for } k=0, 1, 2$$

3 is odd so  
in Case (3)  
of Thm.

0 is a critical, but it is neither local max/min.

②  $f(x) = x^2(2-x^2)$ ,  $x \in [-2, 2]$

Critical points  $f'(p) = 4p - 4p^3 = 0$

$$\Leftrightarrow p(1-p^2) = 0$$

$$\Leftrightarrow p = \pm 1, 0 \leftarrow \text{critical points}$$

Value at critical points:  $f(1) = f(-1) = 1$   
 $f(0) = 0$

Value at boundary points,  $f(2) = f(-2) = -8$

So min at  $p = \pm 2$ ,

max at  $p = \pm 1$ :

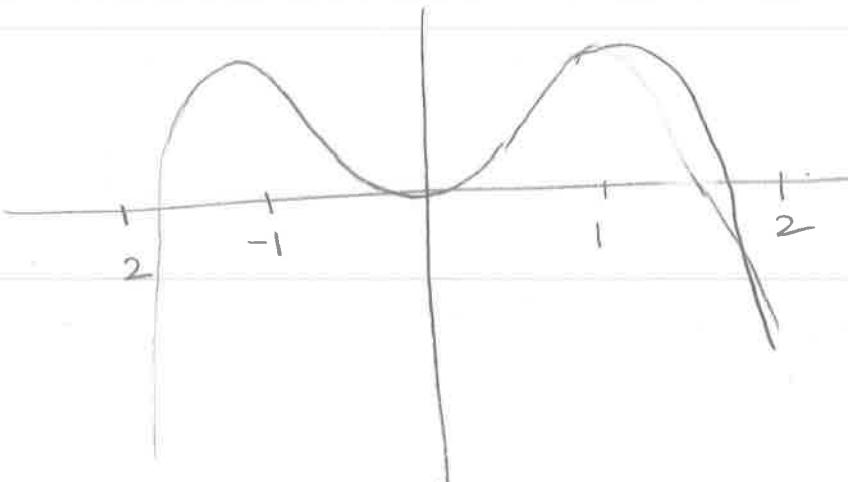
Local max/min

$$f''(p) = 4 - 12p^2$$

$$f''(\pm 1) < 0 \Rightarrow \pm 1 \text{ local max}$$

$$f''(0) > 0 \Rightarrow 0 \text{ local min}$$

2 is even



Concluding remarks: We learn in calculus that diff is a linear approx of the function i.e if  $x \approx p$ , then

$$f(x) \approx f(p) + f'(p)(x-p)$$

Taylor's theorem is a higher order generalizati  
i.e  $x \approx p$ , then

$$f(x) \approx T_{n-1}(x) = f(p) + f'(p)(x-p) + \dots + \frac{f^{(n-1)}(p)}{(n-1)!} (x-p)^{n-1}$$