

## 2.2 METRIC SPACES: COMPACTNESS &

### CONNECTEDNESS

Let  $(X, d)$  be a metric space.

Def<sup>n</sup>: Let  $E \subseteq X$ . A cover of  $E$  is a collection of sets  $\{U_\alpha\}_{\alpha \in I}$  s.t

$$E \subseteq \bigcup_{\alpha \in I} U_\alpha$$

The cover is called open if  $U_\alpha$  is open.

$$\forall \alpha \in I$$

The cover is called finite (resp. countable) if  $I$  is finite (resp. countable).

A sub-cover is a sub-collection  $\{U_\alpha\}_{\alpha \in J}$

where  $J \subseteq I$  s.t  $E \subseteq \bigcup_{\alpha \in J} U_\alpha$ .

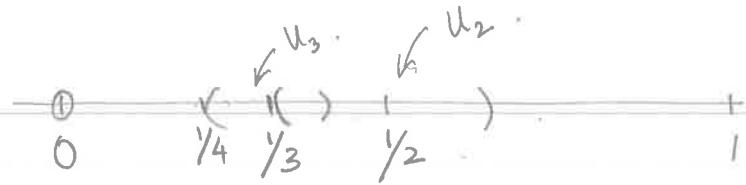
Def<sup>n</sup>: A set  $K \subset X$  is called compact if every open cover of  $K$  has a finite sub-cover.

Example: ) Let  $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ .

Claim:  $K$  is not compact (non-compact).

Define

$$U_k = \left( \frac{1}{k+1}, \frac{k+2}{k(k+1)} \right)$$



Check

1)  $U_k \cap K = \frac{1}{k}$

2)  $K \subset \bigcup_{k=1}^{\infty} U_k$

Clearly no finite sub-collection covers  $K$ . Since

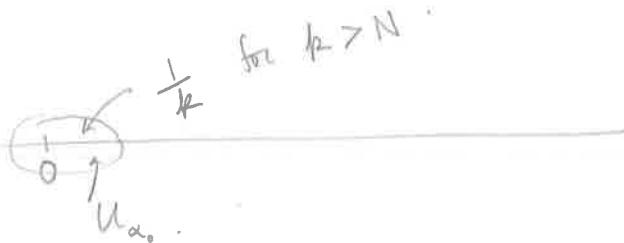
$$\frac{1}{N+1} \notin \bigcup_{k=1}^N U_k$$

$\Rightarrow K$  is not compact.

Claim 2:  $\bar{K} = K \cup \{0\}$  is compact.

PF: Let  $\{U_\alpha\}$  be any open cover. Then  $\exists \alpha_0$  s.t.

$$0 \in U_{\alpha_0}$$



$\exists N$  s.t.  $1/k \in U_{\alpha_0} \forall k > N$ .

For  $k = 1, 2, \dots, N$ , let  $\alpha_k$  s.t.  $1/k \in U_{\alpha_k}$ .

Then

$$K \subset \bigcup_{k=0}^N U_{\alpha_k}$$

So we have a finite sub-cover.

Def<sup>n</sup>: A set  $E \subset X$  is called bounded if  $\exists p \in X$  and  $R > 0$  s.t  $E \subset B_R(p)$ .

Th<sup>m</sup> 2.2.1 Let  $K \subset X$  be compact. Then

- (1) Any infinite subset  $S \subset K$  has a l.p. in  $K$ .
- (2)  $K$  is closed and bounded.

Pf: (1). Sps  $S$  has no l.p. in  $K$ . Then for any  $q \in K$ ,  $\exists r_q > 0$  s.t either

- (1)  $B_{r_q}(q) \cap S = \emptyset$  if  $q \in K \setminus S$ .
- (2)  $B_{r_q}(q) \cap S = \{q\}$  if  $q \in S$ .

In any case  $K \subset \bigcup_{q \in K} B_{r_q}(q)$ .

Each  $s \in S$  belongs to at most one ball. So if there is a finite sub-cover. Then  $S$  would have to be finite. Contradiction!

(2)  $K$  closed: If not, then  $\exists p \in X \setminus K$  s.t  $p$  is a l.p. of  $K$ . Now let  $x_1 \in B_r(p) \cap K$ ,  $x_2 \in B_{r/2}(p) \cap K$ , and so on. That is, if  $x_1, \dots, x_n, \dots$  are picked s.t.

- (a)  $x_n \in B_{r/n}(p) \cap K$
- (b)  $x_n \neq x_k$  if  $k \neq n$ .

(b) can be achieved since  $B_{r/n}(p) \cap K$  is infinite  $\forall n$ .

By (1)  $\Rightarrow \{x_n\}$  has a l.p.  $q \in K$ .

Claim:  $p = q$ .

Pf: If not Then  $r = d(p, q) > 0$ .

Let  $N$  s.t  $y_N < r/2$ .



By construction  $x_k \in B_{y_N}(p) \quad \forall k > N$ .

So  $B_{y_N}(q) \cap K$  can only have finitely many  $x_1, \dots, x_N$ . This contradicts Thm 2.1.6.

K bounded: Fix  $p \in X$ . If  $K$  not bounded  
Then  $\forall n, \exists x_n \in K$  s.t  $d(p, x_n) > n$ .

Claim:  $\{x_n\}$  cannot have a l.p in  $K$ .

Pf: If  $q$  is a l.p.



$B_q(1) \cap \{x_n\}$  has infinite elements.

Let  $x_N \in B_q(1) \cap \{x_n\}$ . s.t  $M > d(p, x_N) + 3$ .  
 $x_M \in B_q(1) \cap \{x_n\}$

Then  $d(p, x_M) \leq d(p, x_N) + d(x_M, x_N)$ .

Since  $x_N, x_M \in B_q(1)$

$$d(x_M, x_N) \leq d(x_M, q) + d(x_N, q) < 2.$$

$$\Rightarrow M \leq d(p, x_M) \leq d(p, x_N) + 2 \leq M + 2.$$

$$\text{But } M \leq d(p, x_N) + 2$$

Contradiction!

Th<sup>m</sup> 2.2.2 Let  $X$  be a compact metric space, and  $E \subset X$  closed. Then  $E$  is compact.

Pf: If not, then  $\exists$  open cover  $\{U_\alpha\}_{\alpha \in I} \text{ s.t. } E \subset \bigcup_{\alpha \in I} U_\alpha$  but no finite sub-collection covers  $E$ . Note that

$$X = \bigcup_{\alpha \in I} U_\alpha \cup E^c$$

$X$  compact  $\Rightarrow \exists \alpha_1, \dots, \alpha_n \text{ s.t.}$

$$X = \bigcup_{k=1}^n U_{\alpha_k} \cup E^c$$

But then  $E \subset \bigcup_{k=1}^n U_{\alpha_k}$  since  $E \cap E^c = \emptyset$ .

So we have extracted a finite sub-cover of  $E$ .

Rk: In general converse to part (b) in Th<sup>m</sup> 2.2.2 need not hold. Consider  $\mathbb{Q}$  as a metric sp. with Euclidean distance.

$$E = \{p \in \mathbb{Q} \mid 2 < p^2 < 3\}$$

is closed & bounded but not compact.

Pf: Step 1:  $k=1$ : let

$$I_n = [a_n, b_n]$$

$$I_n \supset I_{n+1} \Rightarrow a_n \leq a_{n+1}, b_n \geq b_{n+1}$$

$$\begin{array}{ccccccc} & - & [ & & ] & & ] \\ & & a_n & a_{n+1} & & b_{n+1} & b_n \end{array}$$

Clearly  $\{a_n\}$  is upper bounded by  $b_1$ .  
In fact  $a_n \leq b_m \forall n, m$ .

Let  $a^* = \sup a_n$ .

Then  $a^* \geq a_n \forall n$ .

Claim:  $a^* \leq b_n \forall n$ .

If not, then  $\exists b_m \text{ s.t. } b_m < a^*$

$a^* \sup \Rightarrow b_m$  is not an upper bound for  $\{a_n\} \Rightarrow \exists n \text{ s.t. } b_m < a_n$ .

Contradiction!

So  $a^* \in [a_n, b_n] = I_n \forall n \Rightarrow a^* \in \bigcap I_n$ .

Step 2: General  $k$ . Let

$$I_n = \{\vec{x} = (x_1, \dots, x_k) \mid a_{n,j} \leq x_j \leq b_{n,j} \quad j=1, \dots, k\}$$

Define  $I_{n,j} = [a_{n,j}, b_{n,j}]$ ,  $j=1, \dots, k$ .

$$I_{n+1} \subset I_n \Rightarrow I_{n+1,j} \subset I_{n,j} \quad \forall j$$

Def<sup>n</sup>: A subset  $K$  is called limit point compact if every infinite subset  $S \subset K$  has a l.p. in  $K$ .

Th<sup>m</sup> 2.2.3:  $K \subset X$  is compact  $\iff K$  is l.p. compact.

$\Rightarrow$  follows from Th<sup>m</sup> 2.2.1(a). In the next section we show  $\Leftarrow$  when  $X = \mathbb{R}^n$ . The proof of the general case is in the next assignment.

### Compact subsets of $\mathbb{R}^k$

Def<sup>n</sup>: A  $k$ -cell is a subset of  $\mathbb{R}^k$  of the form

$$I = \left\{ \vec{x} = (x_1, \dots, x_k) \mid a_j \leq x_j \leq b_j, j = 1, \dots, k \right\}.$$

Th<sup>m</sup> 2.2.4: A  $k$ -cell  $I$  is compact.

Lemma 2.2.5 (Cantor intersection). Let  $\{I_n\}$  be a countable collection of  $k$ -cells s.t.  $I_n \supseteq I_{n+1}$ . Then  $\bigcap_{n=1}^{\infty} I_n$  is non-empty.

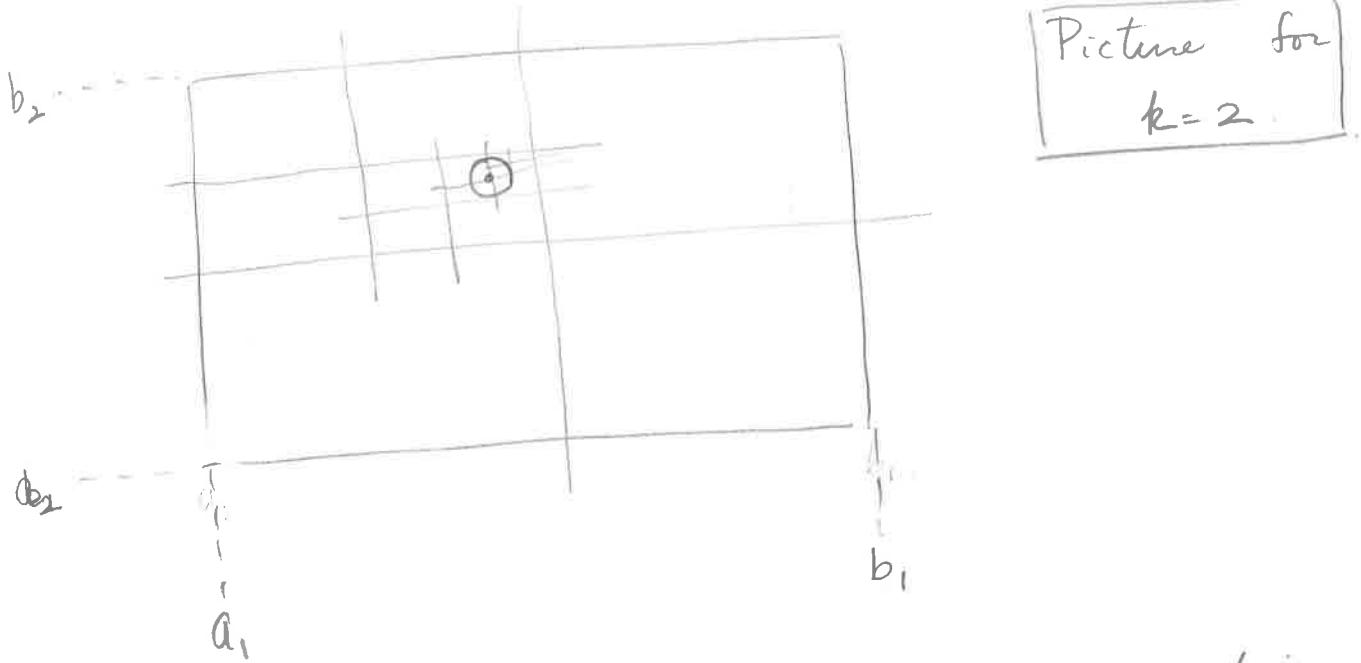
Step 1  $\Rightarrow$  for each  $j$ ,  $\exists t_j \in \bigcap_{n=1}^{\infty} I_{n,j}$ .

Claim:  $\vec{t} = (t_1, \dots, t_k) \in \bigcap I_n$ .

For each  $j, n$ ,  $a_{n,j} \leq t_j \leq b_{n,j}$ .

$\Rightarrow \vec{t} \in I_n \forall n$ .

Pf of Th<sup>m</sup> 2.2.4 Sps  $I$  is not compact. Then there is an open cover  $\{G_\alpha\}_{\alpha \in I}$  s.t. no finite sub-collection covers  $I = [a_1, b_1] \times \dots \times [a_k, b_k]$



Subdivide  $I$  into  $2^k$ -cells by dividing each side into 2.

At least one cannot be covered by a finite sub-collection of  $\{G_\alpha\}$ .

Call this cell  $I_1$ .

Now, let  $S = \text{diag}(I)$  i.e

$$S = \left[ \sum_{j=1}^k (b_j - a_j)^2 \right]^{1/2}.$$

Let  $S_1 = \text{diag}(I_1)$ . So  $S_1 = 2^{-1}S$ .

Iterate to obtain  $I_0 = I, I_1, I_2, \dots, I_n, \dots, I$ .

s.t

$$(1) I_0 \supset I_1 \supset I_2 \dots \supset I_n \supset \dots$$

(2)  $I_n$  is not covered by a finite subcollection of  $\{\text{Gr}_\alpha\}$

$$(3) |\vec{x} - \vec{y}| < 2^{-n}S \quad \forall n, \forall \vec{x}, \vec{y} \in I_n$$

Lemma  $\Rightarrow \exists \vec{p} \in \bigcap_{n=0}^{\infty} I_n$

Since  $\bigcap_{n=0}^{\infty} I_n \subset I_0 = I$ , which is covered by  $\{\text{Gr}_\alpha\}$ ,  $\exists \alpha \in I$  s.t.  $\vec{p} \in \text{Gr}_\alpha$ .

$\text{Gr}_\alpha$  open  $\Rightarrow \exists r > 0$  s.t.  $B_r(\vec{p}) \subset \text{Gr}_\alpha$ .

If n is chosen big enough  $\forall \vec{x} \in I_n$ ,  
since  $\vec{p} \in I_n \Rightarrow |\vec{x} - \vec{p}| < 2^{-n}S < r$ .

$$\Rightarrow I_n \subset B_r(\vec{p}) \subset G_\alpha.$$

So  $G_\alpha$  covers  $I_n$  contradicting 2).

Th<sup>m</sup> 2.2.5  $K \subset \mathbb{R}^k$ . The foll are equivalent  
(TFAE)

- 1)  $K$  is closed and bounded
- 2)  $K$  is compact
- 3)  $K$  is limit point compact.

Pf: 1)  $\Rightarrow$  2).  $K$  bounded  $\Rightarrow K \subset I$  for some  $k$ -cell  $I$  compact,  $K$  closed  $\xrightarrow{\text{Th}^m 2.2.2} K$  is compact.

2)  $\Rightarrow$  3) Th<sup>m</sup> 2.2.1(1)

3)  $\Rightarrow$  1) The proof of Th<sup>m</sup> 2.2.1(2) used only limit point compactness & not compactness.

Cor 2.2.6 (Bolzano - Weierstrass): Every infinite bounded set<sup>S</sup> in  $\mathbb{R}^k$  has a l.p in  $\mathbb{R}^k$ .

Pf: SC I for some  $k$ -cell  $I$ .  $I$  is l.p compact. So Cor follows.

## Connected Sets

Defn:  $A, B \subseteq X$  are called separated if  $A, B$  are non-empty &  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ .

A subset  $E \subset X$  is called connected if  $E$  is NOT the union of two separated sets. Else called disconnected.

Example:  $\mathbb{Q} \subset \mathbb{R}$  is disconnected.

Consider

$$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$$

$$B = (\sqrt{2}, \infty) \cap \mathbb{Q}$$

$$\bar{A} = (-\infty, \sqrt{2}], \quad \bar{B} = [\sqrt{2}, \infty)$$

But  $A \cap \bar{B} = (-\infty, \sqrt{2}) \cap \mathbb{Q} \cap [\sqrt{2}, \infty) = \emptyset$ .

$$B \cap \bar{A} = \emptyset$$

and  $\mathbb{Q} = A \cup B$ .

So  $\mathbb{Q}$  disconnected

Th<sup>m</sup> 2.2.7  $E \subset \mathbb{R}$  is connected  $\iff \forall x, y \in E$   
 $\exists x < z < y$ , then  $z \in E$ .

That is, the only connected subsets of  $\mathbb{R}$  are intervals.

Pf:  $\Rightarrow$ : Sps  $x, y \in E$ ,  $x < z < y$  but  $z \notin E$ .

Consider

$$A = E \cap (-\infty, z)$$

$$B = E \cap (z, \infty)$$

$A, B$  are non-empty since  $x \in A, y \in B$ .

Moreover  $\bar{A} \subset (-\infty, z]$

$$\bar{B} \subset [z, \infty)$$

So  $\bar{A} \cap B = \bar{B} \cap A = \emptyset$ .

So  $A, B$  separated, but  $E = A \cup B$ .

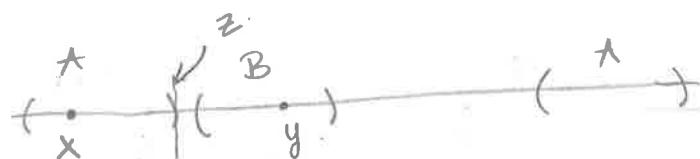
So  $E$  disconnected. Contradiction.

$\Leftarrow$  Sps  $E$  is disconnected. Then  $E = A \cup B$

where  $A$  &  $B$  are separated.

Let  $x \in A$ ,  $y \in B$ . Without loss of generality let

$x < y$ . Then define  $\boxed{z = \sup(A \cap [x, y])}$ .



Clearly  $z \in \overline{A}$ . But then  $z \notin B$  since  $A, B$  separate. In particular  $x \leq z < y$ . Then either  $x = z$  or  $x < z < y$ . In each case  $z \in A$ . In the first case this is because  $x \in A$ . In the second case by hypothesis  $z \in E$ . But  $z \notin B$   
 $\Rightarrow z \in A$ .

Since  $z \in A \Rightarrow z \notin \overline{B}$ .  $\overline{B}$  is closed. So

$$\text{(1)} \quad z \in (B, y)$$

Then  $\exists r > 0$  s.t.  $(z-r, z+r) \cap \overline{B} = \emptyset$ . i.e.  $\exists z_1 \notin \overline{B}$  s.t.  $z < z_1 < y$ .  $z, y \in E \Rightarrow z_1 \in E$ . Clearly  $z_1 \notin B$ . So  $z_1 \in A$ . Then  $z_1 \in A \cap [x, y]$ , and  $z_1 > z$ . contradiction. the fact that  $z = \sup(A \cap [x, y])$ .

Thm 2.2.8:  $(X, d)$  is connected  $\iff$  the only sets that are both open & closed are  $X$  &  $\emptyset$ .

Pf:  $\Rightarrow$ . Sps  $A \subset X$  is both open & closed. Then  $B = A^c$  is also open & closed.

Clearly  $X = A \cup B$ . Also, since  $A$  is closed,

$$\bar{A} = A. \text{ So } \bar{A} \cap B = A \cap B = \emptyset$$

$$\text{Similarly } \bar{B} \cap A = \emptyset.$$

But  $X$  is connected  $\Rightarrow$  either  $A$  or  $B = \emptyset$   
 $\Rightarrow A = X$  or  $\emptyset$ .

$\Leftarrow$  Sps  $X$  is not connected Then  $X = A \cup B$ .  
where  $\bar{A} \cap B = \bar{B} \cap A = \emptyset \& A, B \neq \emptyset$ .

Claim:  $A$  and  $B$  are both closed

Pf:  $A \subset \bar{A}$ . but since  $\bar{A} \cap B = \emptyset \& X = A \cup B$

$$\Rightarrow A = \bar{A}$$

$$\Rightarrow A \text{ is closed}$$

Similarly  $B$  is closed & claim is proved.

Now, since  $B$  is closed  $\Rightarrow B^c = A$  is open.

So  $A$  is both open & closed. But then

$$A = X \text{ or } \emptyset \Rightarrow \text{either } A \text{ or } B = \emptyset$$

Contradiction!

Rk: Let  $(X, d)$  metric space and  $Y \subset X$  with induced metric  $d_Y$ . We have seen that a subset  $U \subset Y$  might be open in  $Y$  but not in  $X$ .

e.g.  $X = \mathbb{R}$ ,  $Y = \mathbb{Q}$ ,  $U = \{x \in \mathbb{Q} \mid 2 < x < 3\}$ .

Then  $U$  is not open in  $\mathbb{R}$ .

On the other hand if  $E \subset Y$ , then one can show that

$E$  is compact (resp. connected) in  $Y$ .

$\Leftrightarrow E$  is compact (resp. connected) in  $X$ .

So we can just say that  $(K, d_K)$  is a compact (resp. connected) metric space.