

0. CARDINALITY OF SETS

• Notation . $\mathbb{N} = \{1, 2, 3, \dots\}$

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{N}, (p, q) = 1\}$$

\mathbb{R} = reals #s

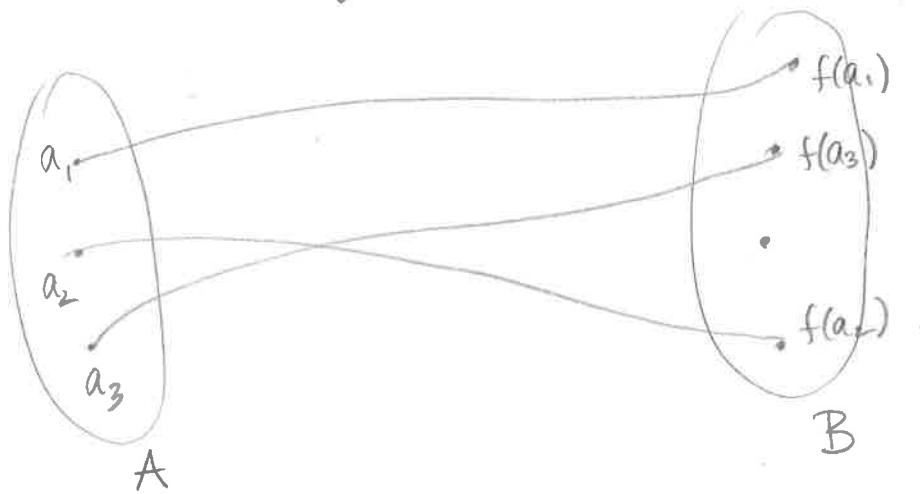
\mathbb{C} = complex #s

$$\mathbb{Z}_n = \{1, 2, 3, \dots, n\}$$

• Functions between sets .

Defⁿ: A, B two sets. A mapping or function $f: A \rightarrow B$ is rule that assigns to every $x \in A$ a unique $y \in B$. We then write

$$y = f(x)$$



A is called domain and

$$f(A) := \{f(x) \mid x \in A\}$$

is called the range or image.

For $C \subseteq B$, the pre-image $f^{-1}(C)$ is defined as

$$f^{-1}(C) = \{x \in A \mid f(x) \in C\}.$$

Defⁿ: A mapping $f: A \rightarrow B$ is

1) Injective or 1-1 if $\forall x, y \in A$

$$f(x) = f(y) \Rightarrow x = y.$$

(or eq $x \neq y \Rightarrow f(x) \neq f(y)$)

2) Surjective or onto if $f(A) = B$. i.e.

$\forall y \in B$, $\exists x \in A$ s.t $f(x) = y$.

3) Bijection if 1-1 AND onto.

Defⁿ: We say two sets have the same cardinality, and write $A \sim B$ if there exists a bijection $f: A \rightarrow B$. Can then define $f^{-1}: B \rightarrow A$ which is also bij.

Defⁿ: A set A is

1) Finite if $A \sim \mathbb{Z}_n$ for some n . We then write $n = |A|$.

2) Countable if $A \sim \mathbb{N}$.

- 3) At most countable if finite or countable.
 4) Uncountable otherwise.

Examples: 1) Integers are countable. Consider
 $f: \mathbb{Z} \rightarrow \mathbb{N}$,

$$f(n) = \begin{cases} 2n & , n \geq 0 \\ 2(-n)-1 & , n \leq 0 \end{cases}$$

Show: f is 1-1 & onto.

2) Even numbers (\mathbb{E}) are countable.

$f: \mathbb{E} \rightarrow \mathbb{N}$ defined by

$$f(n) = n/2.$$

Defⁿ: A sequence of elements in a set A is a function $f: \mathbb{N} \rightarrow A$. We then call

$$f(n) = x_n \in A$$

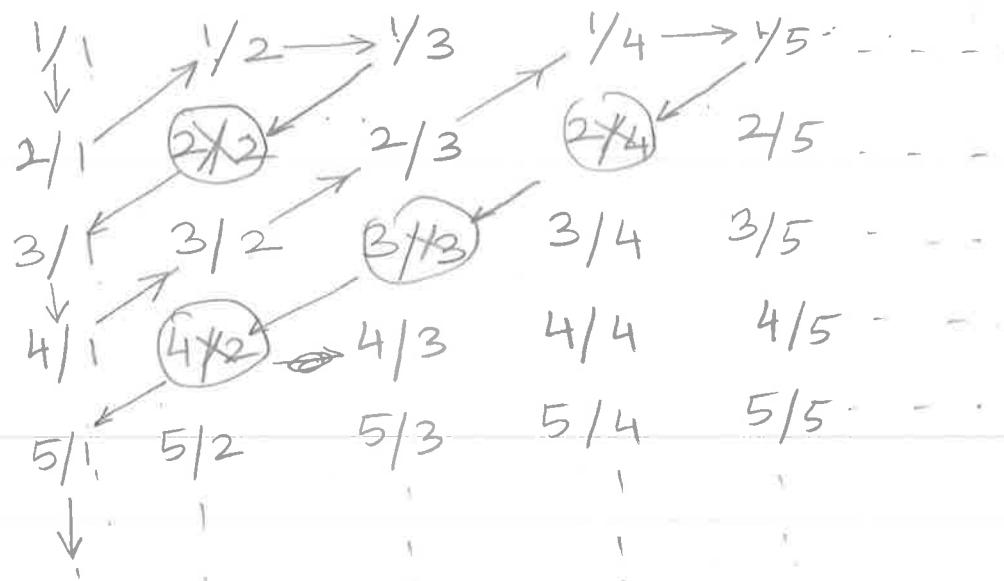
the n^{th} element of the seqⁿ and denote the seqⁿ by $\{x_n\}$.

Rk: A is countable \Leftrightarrow its' elements can be listed as a seqⁿ s.t no two terms are identical

Th^m 0.1 \mathbb{Q} is countable.

"Pf": Step 1: $\mathbb{Q}_{>0} = \{ p \in \mathbb{Q} \mid p > 0 \}$ is countable.

Arrange elements of $\mathbb{Q}_{>0}$ in an array.



So $\mathbb{Q}_{>0} = \{ 1, 2, 1/2, 1/3, 3, 4, 3/2, 2/3, 1/4, 1/5, \dots \}$.

Fact: This exhausts $\mathbb{Q}_{>0}$ (read about).

Step 2: \mathbb{Q} is countable. Sps $f: \mathbb{Q}_{>0} \rightarrow \mathbb{N}$ is the bijection from Step 1. Define

$$F: \mathbb{Q} \rightarrow \mathbb{Z} \quad f(p/q) \quad , \quad p > 0$$

$$F(p/q) = \begin{cases} f(p/q) & , \quad p > 0 \\ 0 & , \quad p = 0 \\ -f(p/q) & , \quad p < 0 \end{cases}$$

Claim: F is 1-1.

Pf: If $F(p/q) = F(s/t)$, \Rightarrow either both $p/q, s/t > 0$ or $p/q, s/t < 0$ or $p, s = 0$. since F preserves sign. But sps $p/q, s/t > 0$ then $f(p/q) = f(s/t)$. Since f is 1-1 $\Rightarrow p/q = s/t$. Similar argument for when $p/q, s/t < 0$ or $p = s = 0$.

Claim: F is surjective

Pf: Follows from f being surj. onto \mathbb{N} .

Similar arg. to step 1 can prove.

Th^m 0.2: Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of countable sets. Then

$$E = \bigcup_{n=1}^{\infty} E_n$$

is countable.

Pf: See Rudin Th^m 2.12.

• Uncountability of \mathbb{R} :

Binary expansion: Every $x \in (0, 1)$ can be written as

$$x = \frac{d_1}{2} + \frac{d_2}{2^2} + \dots = \sum_{k=1}^{\infty} \frac{d_k}{2^k}, \quad d_k = 0 \text{ or } 1$$

We write $x = (0 \cdot d_1 d_2 \dots)_2$.

Expansion need not be unique. For instance.

$$(0 \cdot 1)_2 = (0 \cdot 0111 \dots)_2$$

Since $(0 \cdot 0111 \dots)_2 = \frac{1}{4} + \frac{1}{2^3} + \dots$

$$= \frac{1}{4} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots \right)$$
$$= \frac{1}{4} \left(\frac{1}{1 - \frac{1}{2}} \right) = \frac{1}{2} = (0 \cdot 1)_2$$

CONVENTION: We do not allow binary expansion ending with all zeroes.

So if $(0 \cdot d_1 \dots d_N)_2 = (0 \cdot d_1 \dots d_{N-1} 0111 \dots)_2$.

$d_N = 1$. Then binary exp. are unique.

Th^m 0.4: $[0, 1]$ is an uncountable set.

Pf: If not, then we could list all elements as a seqⁿ $\{x_1, x_2, \dots\}$. Consider their binary exps (with above conv.).

$$x_1 = (0 \cdot d_{11} d_{12} d_{13} \dots)_2$$

$$x_2 = (0 \cdot d_{21} d_{22} d_{23} \dots)_2$$

$$x_3 = (0 \cdot d_{31} d_{32} d_{33} \dots)_2$$

Look at the diagonal.

Define

$$x = (0 \cdot e_1 e_2 \dots)_2$$

where

$$e_k = \begin{cases} 0 & \text{if } d_{kk} = 1 \\ 1 & \text{if } d_{kk} = 0 \end{cases}$$

Then $x \in (0, 1]$. But $x \notin \{x_k\}$ since x differs from x_k in the k^{th} pos of binary exp. Contradiction!

Rk: Argument is called Cantor diagonalization

Lemma 0.5: Let A, B be sets

- 1) Sps $\exists f: A \rightarrow B$ injective & B is countable $\Rightarrow A$ is at most countable.
- 2) Sps $\exists f: A \rightarrow B$ surjective & A countable, then B is at most countable.

Cor 0.6: \mathbb{R} is uncountable.

Pf: Consider $i: (0, 1] \rightarrow \mathbb{R}$

$$i(x) = x$$

i is inj. If \mathbb{R} is countable by (i) above $(0, 1]$ is countable, contradicting Th^m 0.4.
 $\Rightarrow \mathbb{R}$ is uncountable.

Pf of Lem 0.5.

1). Let $B = \{b_1, b_2, \dots\}$, and s.p.s A is infinite

let

$$k_1 = \min \{k \mid b_k \in f(A)\}.$$

$$k_2 = \min \{k > k_1 \mid b_k \in f(A)\}.$$

$$k_n = \min \{k > k_{n-1} \mid b_k \in f(A)\}.$$

Define $a_n = f^{-1}(b_{k_n})$.
Since f is 1-1, f^{-1} exists and is inverse.

Claim: $A = \{a_n\}$.

Clearly $\{a_n\} \subseteq A$. Let $a \in A$. Then $\exists k \in \mathbb{N}$ s.t. $f(a) = b_k$. By defⁿ, $\exists m$ s.t. $k = k_m$. i.e. $f(a) = b_{k_m} \Rightarrow a = f^{-1}(b_{k_m}) \Rightarrow a = a_m$. So $a \in \{a_n\}$.

2). Let $A = \{a_1, \dots\}$. For $b \in B$ let

$$g(b) := \min \{n \mid f(a_n) = b\}$$

Then $g: B \rightarrow \mathbb{N}$ is a function.

Claim: g is injective.

Pf: If $g(b_1) = g(b_2)$, then $f(a_{g(b_1)}) = b_1$ & $f(a_{g(b_2)}) = b_2$.

Since f is a function $\Rightarrow b_1 = b_2$.

$\Rightarrow g$ is injective.

But then by (1), since \mathbb{N} is countable.
(by def^{n!}). $\Rightarrow B$ is at most countable.

