

Arzela - Ascoli Theorem

Ques: Given a seqⁿ of functions $f_n: E \rightarrow \mathbb{R}$, $E \subset \mathbb{R}^m$; when is there a sub-sequence $\{f_{n_k}\}$ s.t $f_{n_k} \xrightarrow{u.c} f$ on E ?

Example: $f_n(x) = x^n$ on $(0, 1)$.

Claim: No sub-seqⁿ converges uniformly.

Pf: If $f_{n_k} \xrightarrow{u.c} f$, then $f(x) \equiv 0$.

$$\text{But } M_{n_k} = \sup_{(0,1)} |f_{n_k} - f| = \sup_{x \in (0,1)} x^{n_k} = 1$$

So $M_{n_k} \not\rightarrow 0$ Contradiction!

We need to introduce two concepts.

Defⁿ: A family of functions F on $E \subset \mathbb{R}^m$ is said to be

(1) pointwise bounded: if $\forall x \in E, \exists M(x) > 0$

$$\text{s.t } |f(x)| < M(x) \quad \forall f \in F$$

(2) uniformly bounded: if $\exists M > 0$ s.t

$$|f(x)| < M \quad \forall x \in E, \forall f \in F$$

Examples 1) $\tilde{F} = \{f_n(x) = x^n, n=1, 2, \dots\}$ on $E = (0, 1)$ (2)

Then $|f_n(x)| \leq 1 \quad \forall f_n \in \tilde{F}, \forall x \in (0, 1)$.

So \tilde{F} is uniformly bounded family.

2) $\tilde{F} = \{f_n: [0, 1] \rightarrow \mathbb{R} \mid f_n(x) = nx\}$

For each x , $\lim_{n \rightarrow \infty} f_n(x) = \infty$, so \tilde{F} is not pointwise bounded.

3) $\tilde{F} = \{f_n: [0, 1] \rightarrow \mathbb{R} \mid f_n(x) = n^2 x^n (1-x)\}$

For each x , $\lim_{n \rightarrow \infty} f_n(x) = 0$

So f_n is pointwise bounded. On the other hand, one can show that the max of f_n is at $x_n = n/n+1$.

$$\begin{aligned} f_n\left(\frac{n}{n+1}\right) &= n^2 \cdot \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \\ &= n \cdot \left(\frac{n}{n+1}\right)^{n+1} = n \cdot \left(1 - \frac{1}{1+n}\right)^{n+1} \\ &\xrightarrow{n \rightarrow \infty} \infty \quad \text{since } \lim_{n \rightarrow \infty} \left(1 - \frac{1}{1+n}\right)^{n+1} = e^{-1} \end{aligned}$$

So \tilde{F} is NOT uniformly bounded.

Th^m 7.1: Let $E \subset \mathbb{R}^m$. If $f_n \xrightarrow{u.c} f$ on E and each f_n is bounded. Then $\{f_n\}$ is uniformly bounded.

Pf: Assignment 9

Defⁿ: A family of functions is called equicont.

if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$ s.t

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in F$$

$$x, y \in E$$

Rk: In particular each $f \in F$ is uniformly cont.

Examples: $L_M(0,1) = \{f: (0,1) \rightarrow \mathbb{R} \mid |f'(t)| \leq M\}$.

For any $t, s \in (0,1)$, $\exists c$ between s & t s.t

$$f(s) - f(t) = f'(c)(s - t).$$

So $\forall f \in L_M(0,1)$

$$|f(s) - f(t)| \leq M|s - t| \quad \begin{matrix} \leftarrow \\ \text{Lipschitz function} \end{matrix}$$

Given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then

$$|s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon \quad \forall f \in L_M(0,1)$$

$$s, t \in (0,1)$$

④

Th^m 7.2: If $K \subset \mathbb{R}^m$ is compact, and $f_n: K \rightarrow \mathbb{R}$ continuous s.t. $f_n \xrightarrow{uc} f$ on K . Then $\{f_n\}$ is equicontinuous.

Pf: Let $\varepsilon > 0$. $\exists N$ s.t. $\forall x \in K$,

$$|f_n(x) - f_N(x)| < \varepsilon/3 \quad \forall n > N \quad (*).$$

Since $f_n \xrightarrow{uc} f$.

Claim $\exists \delta_N > 0$ s.t. $\forall n \geq N$

$$|x - y| < \delta_N \Rightarrow |f_n(x) - f_n(y)| < \varepsilon.$$

Pf: K compact $\Rightarrow f_N: K \rightarrow \mathbb{R}$ uniformly

cont. So $\exists \delta_N$ s.t.

$$|x - y| < \delta_N \Rightarrow |f_N(x) - f_N(y)| < \varepsilon/3.$$

if $n > N$, and $|x - y| < \delta_N$, then

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| \\ &\quad + |f_N(y) - f_n(y)| \end{aligned}$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Done!

Now for $k=1, \dots, N-1$, let $\delta_k > 0$ s.t

$$|x - y| < \delta_k \Rightarrow |f_k(x) - f_k(y)| < \varepsilon.$$

(5)

Let $\delta = \min(\delta_1, \dots, \delta_N)$. Then

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon \quad \forall n$$

$\Rightarrow \{f_n\}$ equicont.

Rk: Proof works if compactness of K is replaced by each f_n being uniformly cont.

Thm 7.3 (Arzela-Ascoli): Let $K \subset \mathbb{R}^m$ compact and $\{f_n\}$ pointwise bounded and equicont on K . Then

(1) $\{f_n\}$ uniformly bounded.

(2) \exists sub-seqⁿ $\{f_{n_k}\}$ which converges uniformly. The proof has several steps. First we need the foll key observation

Lemma 7.4: Let E be a countable set, and $f_n: E \rightarrow \mathbb{R}$ a pointwise bounded seqⁿ. Then \exists sub-seqⁿ f_{n_k} converging pointwise on E .

Pf: Let $E = \{P_1, P_2, \dots\}$. Consider the seqⁿ $\{f_n(P_i)\}$. This is a bounded seqⁿ in \mathbb{R} . So there is a sub-seqⁿ, which we denote by $\{f_{n_k}\}$ s.t

$\{f_{1,k}(p_1)\}$ converges.

Now consider $\{f_{1,k}(p_2)\}$. This is again a bounded seqⁿ. So \exists sub-seq $\{f_{2,k}\}$ s.t $\{f_{2,k}(p_2)\}$ converges.

More generally we construct sequences $S_1, S_2,$

$S_1 : f_{1,1}, f_{1,2}, f_{1,3}, \dots$

$S_2 : f_{2,1}, f_{2,2}, f_{2,3}, \dots$

$S_3 : f_{3,1}, f_{3,2}, f_{3,3}, \dots$

s.t

(1) $S_n \subset S_{n-1}$ i.e S_n is a sub-seqⁿ of S_{n-1} .
 S_1 sub-seqⁿ of $\{f_n\}$.

(2) $\{f_{n,k}(p_n)\}$ converges as $k \rightarrow \infty$.

(3) functions appear in the same order as they appear in $\{f_n\}$.

Now, consider the "diagonal"

$D = f_{1,1}, f_{2,2}, \dots, f_{n,n}, \dots$

Clearly D is a sub-seqⁿ of f_n .

Claim: $\{f_{n,n}(p_m)\}$ converges as $n \rightarrow \infty$ $\forall p_m \in E$.

Pf: Given p_m , note that $\{f_{n,n}\}_{n \geq m}$ is a sub-seqⁿ of S_m . Since $f_{m,k}(p_m) \hookrightarrow$ as $k \rightarrow \infty$
 $\Rightarrow \{f_{n,n}(p_m)\}$ converges as $n \rightarrow \infty$.

Proof of Arzela - Ascoli

(1) $\{f_n\}$ equicont. So $\exists \delta > 0$ s.t.

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < 1. \quad \forall n.$$

Now, $K \subset \bigcup_{p \in K} B_\delta(p)$. But K compact

$\Rightarrow \exists p_1, \dots, p_l$ s.t. $K \subset \bigcup_{i=1}^l B_\delta(p_i)$

i.e. $\forall x \in K$, $\exists p_i \in K$ s.t. $|x - p_i| < \delta$.

Now, f_n pointwise bdd. $\Rightarrow \forall i = 1, \dots, l$, $\exists M_i$

s.t. $|f_n(p_i)| < M_i \quad \forall n$.

Now, if $x \in K$, $n \in \mathbb{N}$, and $p_i \in K$ s.t. $|x - p_i| < \delta$,

$$\begin{aligned} |f_n(x)| &\leq |f_n(x) - f_n(p_i)| + |f_n(p_i)| \\ &\leq 1 + M_i \end{aligned}$$

So if $M = 1 + \max(M_1, \dots, M_l)$. Then $\forall x \in K$,

$$\forall n, \quad |f_n(x)| < M.$$

$\Rightarrow f$ is uniformly bounded.

(2) Let $E \subset K$ be a countable, dense subset.

$$\text{e.g. } E = \{(x_1, \dots, x_m) \in K \mid x_j \in \mathbb{Q}\}.$$

Lemma \Rightarrow \exists sub-seq f_{n_k} s.t. $\{f_{n_k}(x)\}$ converges

$$\forall x \in E. \text{ Let } g_k = f_{n_k} \& E = \{p_1, p_2, \dots\}.$$

Claim: $\{g_j\}$ converge uniformly on K .

Pf: Let $\varepsilon > 0$. $\exists s > 0$ s.t. $\forall k$, by equicont.

$$|x - y| < s \Rightarrow |g_k(x) - g_k(y)| < \varepsilon/3 \quad (*).$$

Since $E \subset K$ is dense, $K \subset \bigcup_{i=1}^{\infty} B_s(p_i)$

K compact $\Rightarrow \exists p_1, \dots, p_l$ s.t. $K \subset \bigcup_{i=1}^l B_s(p_i)$

Now $g_k(p_i)$ converges for $i=1, \dots, l$. So $\exists N_i$ s.t

if $j, k > N_i$, then

$$|g_j(p_i) - g_k(p_i)| < \varepsilon/3.$$

Let $N = \max(N_1, \dots, N_l)$. Then $\forall j, k > N \& \forall i=1, \dots, l$

$$|g_j(p_i) - g_k(p_i)| < \varepsilon/3 \quad (**)$$

Now, if $x \in K$, $\exists i \in \{1, \dots, l\}$ s.t. $|x - p_i| < s$.

$$(*) \Rightarrow |g_j(x) - g_k(x)| < \varepsilon/3 \quad \forall k$$

(9)

If $j, k > N$, then

$$\begin{aligned} |g_j(x) - g_k(x)| &\leq |g_j(x) - g_j(p_i)| + |g_j(p_i) - g_k(p_i)| \\ &\stackrel{(**)}{\leq} \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

So $\{g_k\}$ is uniformly Cauchy in K .

$\Rightarrow g_k$ uniformly converges.

But $\{g_k\}$ is a sub-seqⁿ of f_n , and this proves the theorem.

• Compactness in $C^0(K)$. $K \subset \mathbb{R}^m$, compact.

Recall (from assignment)

$$C^0(K) = \{f: K \rightarrow \mathbb{R} \mid f \text{ cont.}\}.$$

K compact \Rightarrow each f is bounded on K .

Define

$$d(f, g) = \sup_{x \in K} |f(x) - g(x)|.$$

Then

Th^{m7.5}: $(C^0(K), d)$ is a complete metric space.

Moreover (1) $f_n \xrightarrow{d} f \Leftrightarrow f_n \xrightarrow{u.c} f$ on K .

(2) $\{f_n\}$ bdd. in $C^0(K) \Leftrightarrow \{f_n\}$ uniformly bdd on K .

Ques: What are the compact subsets $F \subset C^0(K)$

Th^m 7.6 (Arzela-Ascoli ver-2) A subset $F \subset C^0(K)$ is compact with respect to the metric $d \iff F$ is closed, bounded and equicont.

Rk: Equicont. is necessary. For instance, consider $K = [0, 1] \subset \mathbb{R}$, and let

$$F = \{f \in C^0[0, 1] \mid |f(t)| \leq 1 + t\}.$$

Clearly F is closed & bounded. Also $f_n(x) = x^n \in F$. But no subsequence converges, and hence F is NOT compact.

Pf of Th^m \Rightarrow If F is compact, then it is clearly closed and bounded. To show equicont. let $\epsilon > 0$. Since F is compact, $\exists f_1, \dots, f_N \in F$

$$\text{s.t. } F \subset \bigcup_{j=1}^N B_{\epsilon/3}(f_j)$$

i.e. for any $f \in F$, $\exists f_j \in F$ s.t

$$d(f, f_j) < \epsilon/3 \quad x \in K.$$

$$\Leftrightarrow |f(x) - f_j(x)| < \epsilon/3 \quad \forall x \in K. \quad (*)$$

Each $f_j \in C^0(K)$ and hence is uniformly cont. (11)

So, $\exists \delta_j > 0$ s.t

$$|x - y| < \delta_j \implies |f_j(x) - f_j(y)| < \varepsilon/3. (**)$$

$x, y \in K$

Let $S = \min(\delta_1, \dots, \delta_N)$. For $f \in F$ and $|x - y| < S$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| \\ &\quad + |f_j(y) - f(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

So F is equicont. family.

\Leftarrow : Sps F is closed, bounded and equicont. Let

$\{f_n\}$ be a seqⁿ in F . Since $\{f_n\}$ is bounded & equicont $\xrightarrow[\text{Ascoli}]{\text{Arzela}}$ $\exists f \in C^0(K)$ & sub-seqⁿ f_{n_k} .

s.t $f_{n_k} \xrightarrow{\text{u.c.}} f \Leftrightarrow f_{n_k} \xrightarrow{\text{d}} f$.

F closed $\rightarrow f \in F$.

So every seqⁿ in F has a l.p in F

$\Rightarrow F$ is compact.

Application to differential equation

Consider the initial value prob. (IVP).

$$\begin{cases} f'(x) = \sin(f(x)) \\ f(0) = 1 \end{cases} \quad (*)$$

where say $f \in C^1[0, 1] \subset C^0[0, 1]$.

Of course we know the solⁿ: $f(x) = e^x$.

In many cases, it is impossible to explicitly write down solⁿs, especially to non-linear IVPs, and so existence is shown via setting up an iteration and showing the iteration converges. Here Ascoli-Arzela is useful. We illustrate it in this simple case.

Step 1. Iteration

• Let $f_0 \equiv 1$ on $[0, 1]$. Given f_1, \dots, f_{n-1} , let

$$f_n(x) = \int_0^x \sin(f_{n-1}(t)) dt + 1 \quad (***)$$

• Each $f_n \in C^1[0, 1]$, $f_n(0) = 1$, $|f_n(x)| \leq 1 + x$.

$$f'_n(x) = \sin(f_{n-1}(x))$$

and so $|f'_n(x)| \leq 1 \quad \forall x \in [0, 1]$.

Claim $\{f_n\}$ is equicont.

Pf: Let $\epsilon > 0$, $S = \epsilon$. Then for any $s, t \in [0, 1]$.

$\exists c$ between them s.t

$$f_n(s) - f_n(t) = f'_n(c)(s-t)$$

$$\Rightarrow |f_n(s) - f_n(t)| \leq |f'_n(c)| |s-t| \leq |s-t|$$

So $|s-t| < S \Rightarrow |f_n(s) - f_n(t)| < S = \epsilon$. $\forall n$.

So $\{f_n\}$ is uniformly bounded, and equicont.

Arzela-Ascoli $\Rightarrow \exists$ sub-seqⁿ f_{n_k} s.t

$$f_{n_k} \xrightarrow{u.c} f \in C^0[0, 1].$$

But then $\sin(f_{n_k}) \xrightarrow{u.c} \sin(f)$ (Why?)

So taking limit in (**)

$$f(x) = \int_0^x \sin(f(t)) dt + 1$$

Since $f \in C^0[0, 1]$, $\sin(f(t)) \in C^0[0, 1]$

2nd fundamental theorem $\Rightarrow f$ is diff, and

$$f'(x) = \sin(f(x)).$$

Also $f(0) = 1$. So f solves (*)

Rk: ① This method can be used to show local existence of solⁿ to general 1st order non-linear diff. eq

$$\begin{cases} f' = \varphi(x, f) \\ f(0) = y_0 \end{cases}$$

② Does the whole seqⁿ converge? Here uniqueness of solⁿ plays a role. If (*) has a unique solⁿ then $f_n \xrightarrow{u.s} f$. If not, then numerical approx. can bifurcate to different solutions.