Solutions to Assignment-3

September 19, 2017

1. Let \((X, d)\) be a metric space, and let \(Y \subset X\) be a metric subspace with the induced metric \(d_Y\). Let \(E \subset Y\).

   (a) Show that a set \(U \subset Y\) is open in \(Y\) if and only if there is a subset \(V \subset X\) open in \(X\) such that \(U = V \cap Y\). As an example, consider \(X = \mathbb{R}, Y = [0, 1]\). Then \(U = [0, 1/2)\) is an open subset of \(Y\) with the induced metric. In this case we can take \(V = (-1, 1/2)\). Then \(V\) is open in \(\mathbb{R}\) and \(U = Y \cap V\).

   **Solution:** We will denote the balls of radius \(r\) around \(p \in Y\) with respect to metrics \(d_Y\) and \(d\) by \(B^Y_r(p)\) and \(B^X_r(p)\) respectively. That is
   \[B^Y_r(p) = \{y \in Y \mid d(p, y) < r\}, \quad B^X_r(p) = \{x \in X \mid d(p, x) < r\}\]

   The key observation is that
   \[B^Y_r(p) = B^X_r(p) \cap Y.\]

   As an illustration take \(X = \mathbb{R}\) and \(Y = \mathbb{Q}\). Then \(B^\mathbb{Q}_1(0)\) consists of all the rationals in the interval \((-1, 1)\) while \(B^\mathbb{R}_1(0)\) is the whole interval \((-1, 1)\) and clearly \(B^\mathbb{Q}_1(0) = B^\mathbb{R}_1(0) \cap \mathbb{Q}\).

   Coming back to the problem, suppose \(U \subset Y\) is open (with respect to the subspace metric \(d_Y\)). Then for each \(p \in U\), there exists an \(r_p > 0\) such that \(B^Y_{r_p}(p) \subset U\). So we can write
   \[U = \bigcup_{p \in U} B^Y_{r_p}(p).\]

   Let \(V = \bigcup_{p \in U} B^X_{r_p}(p)\). Then clearly \(V\) is open in \(X\) since arbitrary union of open sets is open, and by the above property of balls, it is clear that \(U = V \cap Y\).

   For the converse, suppose \(U = V \cap Y\) where \(V\) is open in \(X\). Let \(p \in U\). Then \(p \in V\), and since \(V\) is open in \(X\), there exists \(r_p > 0\) such that \(B^X_{r_p}(p) \subset V\). But then \(B^Y_{r_p}(p) = B^X_{r_p}(p) \cap Y \subset U\).

   This shows that for any \(p \in U\) we have a ball \(B^Y_{r_p}(p) \subset U\), and so \(U\) is open in \(Y\).

   (b) Show that \(E\) is compact subset of \(Y\) (with respect to the metric \(d_Y\)) if and only if it is a compact subset of \(X\) (with respect to the metric \(d\)).

   **Solution:** Suppose \(E \subset Y \subset X\) is a compact subset of \(Y\) (with respect to \(d_Y\)). Let \(\{V_\alpha\}\) be a cover of \(E\) by open subsets of \(X\). By the above part, \(U_\alpha = V_\alpha \cap Y\) will be a cover of \(E\) by open subsets of \((Y, d_Y)\). Since \(E\) is compact with the subspace metric, there exists \(\alpha_1, \ldots, \alpha_N\) such that
   \[E \subset \bigcup_{k=1}^N U_{\alpha_k} \subset \bigcup_{k=1}^N V_{\alpha_k}.\]

   So we have managed to extract a finite sub-cover of \(\{V_\alpha\}\). This shows that \(E\) is compact in \((X, d)\). The converse is also similar.
(c) Show that $E$ is a connected subset of $Y$ (with respect to the metric $d_Y$) if and only if it is a connected subset of $X$ (with respect to the metric $d$).

Solution: Again we show one direction leaving the converse as an exercise. Suppose $E \subset Y \subset X$ is a connected subset of $(Y, d_Y)$ but not $(X, d)$. Then there exists non-empty subsets $A$ and $B$ of $X$ such that $E = A \cup B$ but $\overline{A}^X \cap B = \overline{B}^X \cap A = \emptyset$. Here we are taking closures with respect to the metric $d$.

Claim. $\overline{A}^Y \cap B = \overline{B}^Y \cap A = \emptyset$.

Clearly $A \cap B = \emptyset$. Suppose $p \in \overline{A}^Y \cap B$. Then $p$ is a limit point of $A$ (in the metric $d_Y$). So for any $r > 0$, $B_r^Y(p) \cap A \neq \emptyset$. Since $B_r^Y(p) \subset B_r^X(p)$, this shows that $B_r^X(p) \cap A \neq \emptyset$ for any $r > 0$. This shows that $p$ is a limit point of $A$ even in the metric $d$. This is a contradiction since $\overline{A}^X \cap B = \emptyset$. This completes the proof of the claim.

But then we have written $E = A \cup B$ where $A$ and $B$ are non-empty separated sets with respect to the metric $d_Y$, contradicting the fact that $E$ is connected with respect to $d_Y$. Hence $E$ must be connected with respect to the metric $d$ too.

2. A subset $E \subset \mathbb{R}^n$ is called convex if for any two points $p, q \in E$, the straight line $l(t) = (1 - t)p + tq, \ t \in [0, 1]$ joining the two points is contained completely in $E$. Show that any convex set is connected. [Hint. Argue by contradiction.]

Solution: If not, then we can write $E = A \cup B$, where $A$ and $B$ are non-empty and separated, that is, $A, B \neq \emptyset$, and $A \cap B = \overline{A} \cap B = \emptyset$. Recall that this also means that $A$ and $B$ are both open and closed in $E$.

Choose $p \in A$ and $q \in B$. Since $E$ is convex the straight line $l$ joining them is contained in $E$. Intuitively, there will be a first point where the line exits $A$ and enters $B$. This point will lie in $\overline{A} \cap B$, a contradiction. More rigorously, let $T = \sup\{t \mid l(t) \in A \text{ for all } s \leq t\}$.

Then $T$ is the maximum time such that the line is always in $A$ for any time smaller than $T$.

Claim-1. $l(T) \notin A$.

Proof. Now suppose $l(T) \in A$. Since $A$ is open in $E$, there is an $\varepsilon > 0$ such that $\{x \in E \mid |x - l(T)| < \varepsilon\} \subset A$.

Now note that $|l(s) - l(T)| = |(T-s)p + (s-T)q| \leq |s-T||p| + |s-T||q| = |s-T||p| + |q||$.

So if $s \in [T, T + \frac{\varepsilon}{|p|+|q|})$, then $|s-T| < \varepsilon/(|p|+|q|)$, and so $|l(s) - l(T)| < \varepsilon$.

By our choice of $\varepsilon$ and $T$, this shows that $l(s) \in A$ for all $s \in [0, T + \frac{\varepsilon}{|p|+|q|})$, contradicting the maximality of $T$. This proves the claim.

Now, since $E = A \cup B$, the claim implies that $l(T) \in B$. In particular $T > 0$, since $l(0) = p \in A$.

Claim-2. $l(T)$ is a limit point of $A$. 

2
Proof. Let $\varepsilon > 0$. We need to show that there is some $x \in B_\varepsilon(l(T)) \cap A$. Now let

$$s = T - \frac{\varepsilon}{2(|p| + |q|)}.$$

Then by the estimate above,

$$|l(s) - l(T)| \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so $l(s) \in B_\varepsilon(l(T))$. But by definition of $T$, since $s < T$, automatically $l(s) \in A$, hence proving the claim.

So we have now shown that $l(T) \in A \cap B$, a contradiction.

3. A metric space $(X, d)$ is called separable if it has a countable dense subset. A collection of open sets $\{U_\alpha\}$ is called a basis for $X$ if for any $p \in X$ and any open set $G$ containing $p$, $p \in U_\alpha \subset G$ for some $\alpha \in I$. The basis is said to be countable if the indexing set $I$ is countable.

(a) Show that $\mathbb{R}^n$ is countable. Hint. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Solution: Let $\mathbb{Q}^n \subset \mathbb{R}^n$ be the set of all points with all rational coordinates. Then $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, and is countable. So $\mathbb{R}^n$ is separable.

(b) Prove that a metric is separable if and only if it has a countable basis of open sets. Hint. One direction is not hard (which one?). For the other direction, think of smaller and smaller balls of rational radii.

Solution: Proof of $\iff$. Suppose $(X, d)$ has a countable basis $\{U_\alpha\}$. We can assume without loss of generality that none of the basis elements are identical open sets. Pick a point $p_\alpha \in U_\alpha$. Let $P$ be the collection of points $\{p_\alpha\}$. Note that unless $X$ is finite (in which case the proof is trivial anyway), we can always pick points which are not repeated. We claim that $X = P$. To see this, let $x \in X \setminus P$. We have to show that $x$ is a limit point of $P$. Let $G$ be any open set containing $x$. Then by definition of basis, there is a $k$ such that $x \in U_k \subset G$. But then $p_k \in G \cap P$. So every open set around $x$ intersects $P$ at a point different from $x$ itself (since $x \notin P$). This shows that $x$ is a limit point of $P$.

Proof of $\Rightarrow$. Let $S = \{x_1, x_2, \cdots\}$ be the countable dense subset, and let $U_{n,j} = B_{1/n}(x_j)$. We claim that $\{U_{n,j}\}$ forms a basis, and since there are clearly countable number of sets, this completes the proof. To see that it forms a basis, let $x \in X$ and $G$ be an open set containing $x$. Since $G$ is open, there is an integer $N$ such that the ball $B_{4/N}(x) \subset G$. Since $S$ is dense, $x_j \in B_{1/N}(x)$ for some $j$. Choose an integer $N/2 < M < N$ Then we claim that

$$x \in B_{1/M}(x_j) \subset B_{4/N}(x).$$

The first inclusion follows since $M < N$ and $d(x, x_j) < 1/N < 1/M$. For the second inclusion, let $y \in B_{1/M}(x_j)$. Then

$$d(x, y) \leq d(x, x_j) + d(y, x_j) < \frac{1}{N} + \frac{1}{M} < \frac{3}{N}.$$

In particular,

$$x \in U_{M,j} \subset G,$$

completing the proof.

(c) Prove that every compact metric space is separable. Hint. First, cover the metric space by balls of
4. The aim of this exercise is to complete the proof that compactness and limit point compactness are equivalent. Let \( (X,d) \) be a limit point compact metric space.

(a) Show for every \( \delta > 0 \), \( X \) can be covered by finitely many balls of radius \( \delta \). (Note that this is easy for a set already known to be compact; see problem 4 from the previous assignment).

**Solution:** Pick any point \( x_1 \). Then pick \( x_2 \) such that \( d(x_2, x_1) \geq \delta \). If there is no such point then already \( X = B_{\delta}(x_1) \) and the claim is proved with \( N = 1 \). Now inductively, having picked \( x_1, x_2, \cdots, x_{n-1} \) pick an \( x_n \) such that \( d(x_n, x_j) > \delta \) for all \( j = 1, \cdots, n-1 \). We claim that the process terminates in a finitely many steps, thus proving the claim. If not, then we have a sequence \( \{x_n\} \). Since \( X \) is limit point compact, there is a limit point \( p \in X \). So there is an infinitely many terms of the sequence in the ball \( B_{\delta/2}(p) \). If \( x_n \) and \( x_m \) are two such points, then \( d(x_n, x_m) \leq d(x_n, p) + d(x_m, p) \leq \delta/2 + \delta/2 = \delta \) contradicting the fact that \( d(x_n, x_m) > \delta \). So the claim is proved.

(b) If \( F_n \) is a collection of non-empty closed subsets of \( X \) such that \( F_{n+1} \subseteq F_n \) for all \( n \), then show that \( \cap_{n=1}^{\infty} F_n \) is non-empty.

**Solution:** Choose points \( x_n \in F_n \). If the range of the sequence \( \{x_n\} \) is finite, then clearly there is a point, say \( x_N \), which is in infinitely many of the sets \( F_n \). Then, since the sets are decreasing, clearly \( x_N \) will lie in the common intersection. So suppose the range \( \{x_n\} \) is infinite. Then since \( X \) is limit point compact, there is a limit point \( p \).

**Claim.** \( p \in \cap_{n=1}^{\infty} F_n \).

**Proof.** If not, then there exists and \( N \) such that \( p \notin F_N \). Since \( F_N \) is closed, there is a ball \( B_r(p) \) such that \( B_r(p) \cap F_N = \emptyset \). Since \( F_n \subseteq F_N \) for all \( n > N \), clearly \( B_r(p) \cap F_n = \emptyset \) for all \( n > N \). On the other hand, since \( p \) is a limit point, \( B_r(p) \cap \{x_n\} \) has infinitely many points. In particular there is an \( n > N \) such that \( x_n \in B_r(p) \). But \( x_n \in F_n \) which is a contradiction since \( B_r(p) \cap F_n = \emptyset \).

(c) Prove that limit point compactness implies compactness.

**Solution:** The proof is similar to the proof that closed and bounded sets in \( \mathbb{R}^n \) are compact. We proceed by contradiction. Suppose there is an open cover \( \{G_n\} \) such that no finite sub
collection covers \( X \). By the proof of part (b) there exists a finite collection of points \( x_1, \ldots, x_N \) such that \( X \subset \bigcup_{j=1}^{N} B_1(x_j) \). At least one of the balls cannot be covered by a finite sub-collection of \( \{G_\alpha\} \). Label this ball \( B_1 \). Clearly \( \overline{B_1} \) is also limit point compact (Why?), and so we can cover \( B_1 \) with finitely many balls of radius \( 1/2 \) and pick a ball which cannot be covered by a finite sub-collection from \( \{G_\alpha\} \). We continue and obtain a sequence of balls \( \{B_j\} \) such that

- \( B_{j+1} \subset B_j \) for all \( j \).
- The radius of \( B_j \) is \( 1/j \).
- No finite sub-collection of \( \{G_\alpha\} \) covers \( B_j \). In particular, no \( B_j \) is contained in any of the \( G_\alpha \)s.

Now, pick a point \( x_j \in B_j \). If the range \( \{x_j\} \) is finite, then one of the points, say \( x_1 \), belongs to infinitely many of the balls \( B_j \). If \( x_1 \in G_\alpha \), then since \( G_\alpha \) is open, there is a \( j \) big enough so that \( x_1 \in B_j \subset G_\alpha \). But this contradicts property 3 above.

If the range \( \{x_j\} \) is infinite, then by limit point compactness, there exists a limit point \( p \in X \). Let \( \alpha \) such that \( p \in G_\alpha \). Since \( G_\alpha \) is open there is an \( r > 0 \) such that \( B_r(p) \subset G_\alpha \). Moreover, since \( p \) is a limit point of \( \{x_j\} \), there is subsequence \( \{x_{j_k}\} \) such that \( x_{j_k} \in B_{r/2}(p) \). Choose \( j_k \) big enough such that \( 1/j_k < r/2 \). Then by triangle inequality, \( B_{j_k} \subset B_r(p) \subset G_\alpha \) contradicting property 3.