Mid Term-1 : Practice problems

These problems are meant only to provide practice; they do not necessarily reflect the difficulty level of the problems in the exam. The actual exam problems are likely to be easier because of the time constrains.

1. Let $\mathcal{F}$ be a collection of some subsets of $\mathbb{R}^k$, and let $S = \cup_{A\in\mathcal{F}} A$ and $T = \cap_{A\in\mathcal{F}} A$. For each of the following statements, either give a proof or provide a counterexample.
   (a) If p is a limit point of T, then p is a limit point of each $A \in \mathcal{F}$.
   (b) If p is a limit point of S, then p is a limit point of at least one set $A \in \mathcal{F}$.

2. Consider the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by the formula
$$d(\mathbf{x}, \mathbf{y}) = \max(|x_1 - y_1|, |x_2 - y_2|),$$
where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.
   (a) Show that d defines a metric on $\mathbb{R}^2$. This is called the Manhatten metric.
   (b) Draw the ball of radius one centered at the origin in the above metric. What geometric shape do you obtain?
   (c) Are closed and bounded sets compact with this new metric? If so, give a proof. If not, give an example of a closed and bounded set (in this new metric) which is not compact.

3. Let $\{a_n\}$ be a sequence of real numbers such that $|a_n| \leq 2$, and
$$|a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1}^2 - a_n^2|$$
for all $n \geq 1$. Show that a_n converges. **Hint.** Show that the sequence is Cauchy. You might have to use the fact that
$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

4. Let $2^{\mathbb{N}}$ denote the collection of subsets of $\mathbb{N}$. That is,
$$2^{\mathbb{N}} = \{A \mid A \subset \mathbb{N}\}.$$
Show that $2^{\mathbb{N}}$ is uncountable. **Hint.** Proceed by contradiction and use anx argument similar to Cantor diagonalization.

5. Let A and B be disjoint closed subset of a metric space (X, d). Show that there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. **Hint.** First show that there is an open set U containing A such that $\overline{U}$ and B are disjoint.

6. Let (X, d) be a complete metric space.

   (a) Suppose $\{E_n\}$ is a sequence of closed and bounded sets, $E_{n+1} \subset E_n$ for all n, and if

   $$\lim_{n \to 0} \operatorname{diam}(E_n) = 0,$$

   where for any subset E, the diameter is defined as $\operatorname{diam}(E) = \sup_{x,y \in E} d(x, y)$, then $\cap_{n=1}^{\infty} E_n$ consists of **exactly** one point.

   (b) If $\{G_k\}$ is a sequence of dense open subsets of $\mathbb{R}^k$, show that $\cap G_k$ is non-empty. **Hint.** Find a shrinking sequence of neighborhoods E_n such that $\overline{E}_n \subset G_n$ and apply part (a).

   (c) From part (d) show directly that if $\mathbb{R}^k = \cup_{k=1}^{\infty} F_k$ for closed subsets F_k, then at least one of F_k has a non-empty interior.