Assignment-4

(Due 10/03)

1. For any two bounded sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, show that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Give an example of strict inequality.

Note. The inequality also remains true for unbounded sequences provided the right side is not of the form $\infty - \infty$ or $-\infty + \infty$.

2. This exercise shows that even in a complete metric, a closed and bounded set need not be compact. Let

$$l^\infty(\mathbb{C}); = \{\{a_k\}_{k=1}^\infty \mid \exists M > 0 \text{ such that } |a_k| < M, \text{ for all } k\}.$$

That is, $l^\infty(\mathbb{C})$ is the set of all *bounded* sequences of complex numbers. Note that the M will vary from sequence to sequence. For two sequences $A = \{a_n\}$ and $B = \{b_n\}$, define

$$d(A,B) = \sup_k |a_k - b_k|.$$

   (a) For any two sequences $A, B \in l^\infty(\mathbb{C})$, show that $d(A,B)$ is a finite number.

   (b) Let E_n be the sequence with 1 at the n^{th} place and zero everywhere else, and let O be the sequence with zeroes everywhere. What is $d(E_n, O)$? $d(E_n, E_m)$ for $n \neq m$?

   (c) Show that $(l^\infty, \mathbb{C})$ is a *complete metric space*. **Hint.** First show that d satisfies all axioms of a metric. To show completeness, let $A_n = \{a_{nk}\}_{k=1}^\infty$ be a Cauchy sequence in $l^\infty(\mathbb{C})$. First, show that for each fixed k, $\{a_{nk}\}_{n=1}^\infty$ is a Cauchy sequence of complex numbers, and hence by completeness of complex numbers, it converges to a limit b_k. Then show that the sequence $B = \{b_k\}$ is the limit of the sequence A_n in $l^\infty(\mathbb{C})$. It helps to visualize a_{nk} as elements of an infinite array with each row given by the sequence A_n.

   (d) Show that the set $\overline{B_1(O)}$ is closed and bounded, but not compact. **Hint.** Show that the sequence E_n from above has no limit point.

3. The aim of this exercise is to generalize Bolzano-Weierstrass theorem. A metric space X is called *totally bounded* if for any $r > 0$, there is a finite covering of X by balls of radius r. That is, given any $r > 0$, we can find finitely many points $p_1, p_2, \cdots, p_N$ (where N will in general depend on r) such that

$$X = \cup_{k=1}^N B_r(p_k).$$

   (a) Show that a set in $\mathbb{R}^n$ (considered as a metric subspace) is totally bounded if and only if it is bounded.

   (b) A metric space is compact if and only if it is complete and totally bounded.

   Hint. Compactness implies completeness and total boundedness is easy. For the converse, proceed by contradiction, and use an argument similar to the final problem from the previous assignment.