Assignment-3

(Due 09/19)

1. Let (X, d) be a metric space, and let $Y \subset X$ be a metric subspace with the induced metric d_Y. Let $E \subset Y$.
   (a) Show that a set $U \subset Y$ is open in Y if and only if there is a subset $V \subset X$ open in X such that $U = V \cap Y$. As an example, consider $X = \mathbb{R}$, $Y = [0, 1]$. Then $U = [0, 1/2)$ is an open subset of Y with the induced metric. In this case we can take $V = (-1, 1/2)$. Then V is open in $\mathbb{R}$ and $U = Y \cap V$.
   (b) Show that E is compact subset of Y (with respect to the metric d_Y) if and only if it is a compact subset of X (with respect to the metric d).
   (c) Show that E is a connected subset of Y (with respect to the metric d_Y) if and only if it is a connected subset of X (with respect to the metric d).
2. A subset $E \subset \mathbb{R}^n$ is called *convex* if for any two points $p, q \in E$, the straight line
$$\mathbf{l}(t) = (1-t)p + tq, \ t \in [0, 1]$$
joining the two points is contained completely in E. Show that any convex set is connected. **Hint.** Argue by contradiction.
3. A metric space (X, d) is called *separable* if it has a countable dense subset. A collection of open sets $\{U_\alpha\}$ is called a *basis* for X if for any $p \in X$ and any open set G containing p, $p \in U_\alpha \subset G$ for some $\alpha \in I$. The basis is said to be countable if the indexing set I is countable.
   (a) Show that $\mathbb{R}^n$ is separable.
   (b) Prove that a metric space is separable if and only if it has a countable basis of open sets. **Hint.** One direction is not hard (which one?). For the other direction, think of smaller and smaller balls of rational radii.
   (c) Prove that if X is separable, then given any open cover $\{U_\alpha\}_{\alpha \in I}$ of X one can extract a countable sub-cover.
4. In lectures, we have seen that compactness implies limit point compactness. The aim of this exercise is to show the converse, and hence complete the proof that compactness and limit point compactness are equivalent. Let (X, d) be a limit point compact metric space.
   (a) Show for every $\delta > 0$, X can be covered by finitely many balls of radius δ. (Note that this is easy for a set already known to be compact; see problem 4 from the previous assignment).
   (b) If F_n is a collection of non-empty closed subsets of X such that $F_{n+1} \subset F_n$ for all n, then show that $\cap_{n=1}^{\infty} F_n$ is non-empty.
   (c) Prove that X is compact.
   Hint. Apply the argument used in class to prove that k-cells are compact, replacing shrinking cells by shrinking balls, and using part(a).