Assignment-1

(due 09/05)

1. If F is an ordered field, show that for any $x \in F$, $x^2 \geq 0$, and hence prove that no ordering can be defined on the complex pane that turns it into an ordered field.

2. (Cauchy-Schwarz inequality)

   (a) Let $z_1, \cdots, z_n$ and $w_1, \cdots, w_n$ be complex numbers. Show that

   $$\Big|\sum_{j=1}^{n} z_j \bar{w}_j\Big| \leq \Big(\sum_{j=1}^{n} |z_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{n} |w_j|^2\Big)^{1/2}.$$

   When is equality satisfied?

   (b) Now suppose all the numbers are real, and for simplicity, suppose $n = 3$. Can you interpret this inequality using the dot-product for vectors, and give a semi-rigorous geometric proof of the inequality?

3. (Triangle inequality) Recall that for $\vec{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$,

   $$|\vec{v}| = \sqrt{v_1^2 + \cdots v_n^2}.$$

   (a) If $\vec{u}, \vec{v}$ are points in $\mathbb{R}^n$, show that

   $$|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|.$$

   Show that equality is satisfies if and only if $\vec{u}$ and $\vec{v}$ lie on the same ray emanating from the origin.

   (b) More generally, if $\vec{v}_1, \cdots \vec{v}_k \in \mathbb{R}^n$ are k-points, show that

   $$|\vec{v}_1 + \cdots + \vec{v}_k| \leq |\vec{v}_1| + \cdots |\vec{v}_k|.$$

4. This exercise assumes a working knowledge of infinite series from calculus including the formula for the sum of a geometric series. In particular, you might need to use the formulas

   $$\sum_{k=0}^{N} ar^k = a\frac{1 - r^N}{1 - r}$$
   $$\sum_{k=0}^{\infty} ar^k := \frac{a}{1 - r}, \quad \text{if } -1 < r < 1.$$

   The *Cantor set*, which we denote by C is defined in the following way. Let

   $$\begin{aligned} C_0 &= [0, 1], \\ C_1 &= [0, 1/3] \cup [2/3, 1], \\ C_2 &= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \\ &\vdots \\ C &= \cap_{n=0}^{\infty} C_n. \end{aligned}$$

Note that each C_n is a union of finitely many connected intervals, and at each stage C_n is constructed from C_{n-1} by removing the middle thirds from each of the intervals (Draw a picture!).

The aim of this exercise is to show that the Cantor set is uncountable. We will need some knowledge of *ternary expansions*. It is a fact that any number $x \in [0,1]$ can be written as

$$x = \sum_{n=1}^{\infty} \frac{d_k}{3^k},$$

where $d_k = 0, 1$ or 2. We use the notation $x = (0.d_1 d_2 \cdots)_3$. Note that this expansion is not unique. For instance $(0.100000\cdots)_3 = (0.0222222\cdots)_3$ or $(0.2000\cdots)_3 = (0.122222\cdots)_3$. We will say that a ternary expansion is in the *standard form* if it does **NOT** end with a 1 followed by an infinite string of all zeros or all twos. So for instance $(0.0222222\cdots)_3$ is in the standard form while $(0.100000\cdots)_3$ is not, even though they both represent $1/3$. The next two exercises show that every number in $[0,1]$ has exactly one standard form ternary expansion.

(a) Suppose $(0, d_1 d_2 \cdots)_3 = (0.e_1 e_2 \cdots)_3$, and N is the first integer such that $d_N \neq e_N$. Without loss of generality, we can assume that $d_N < e_N$. Then show that $e_N = d_N + 1$ and that $d_k = 2$ for all $k > N$ and $e_k = 0$ for all $k > N$.

(b) Show that any $x \in [0,1)$ has **exactly** one standard form ternary expansion.

(c) Note that the set C_N is the union of 2^N intervals. Show that the **left end point** of each of the intervals is given by $(0.d_1 d_2 \cdots d_N)_3$ where $d_k = 0$ or 2 for all $1 \leq k \leq N$. For instance the left end points of C_2 are given by $0 = (0.00)_3,\ 2/9 = (0.02)_3,\ 2/3 = (0.20)_3,\ 8/9 = (0.22)_3$.

(d) Show that any $x \in C_N$ written out in the standard ternary form $x = (0.d_1 \cdots)_3$ has the property that $d_k = 0$ or 2 for all $k \leq N$.

(e) Show that a number $x \in C$ can be written as $x = (0.d_0 d_1 \cdots)_3$ with $d_k = 0$ or 2 for all $k = 1, 2, \cdots$ in **exactly** one way.

(f) Use this to show that C is an uncountable set.