MATH 104 : Mid-Term-1

26 September, 2017

Name: ____________________

- You have 80 minutes to answer the questions.
- Use of calculators or study materials including textbooks, notes etc. is not permitted.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
- For questions with multiple parts, you can solve a part assuming the previous parts and get full credit for that particular part.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. State whether the following statements are true or false. No explanation needed.

(a) (2 points) Let a_n be a bounded sequence of real numbers. If $L = \limsup_{n\to\infty} a_n$, then there is a subsequence a_{n_k} such that

$$\lim_{k\to\infty} a_{n_k} = L.$$

(b) (2 points) $E \subset X$ is dense if and only if every non-empty open set has a non-empty intersection with E.

(c) (2 points) Let (X, d) be a metric space, and $\{x_n\}$ a sequence in X. Then $X \setminus \{x_1, x_2, \cdots\}$, that is X with the sequence of points removed, is open.

(d) (2 points) The set $[a, b] \cap \mathbb{Q}$ is **both** open and closed in $\mathbb{Q}$ (with the subspace metric induced from $\mathbb{R}$) if and only if a and b are **both** irrational.

(e) (2 points) In a complete metric space, closed and bounded sets are compact.

2. (a) Let $E \subset \mathbb{R}$ be bounded above, so that $\alpha = \sup E$ exists.

   i. (4 points) Show that $\alpha \in \overline{E}$.

   ii. (2 points) Is α always a limit point? If so, prove it. If not, give a counter example.

   (b) (4 points) Let (X, d) be a metric space and let $E \subset X$ be any subset. Show that for any $r > 0$, the r-neighborhood of E defined by

$$T_r(E) := \{x \mid d(x, p) < r \text{ for some } p \in E\},$$

   is an open subset of X.

3. The function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|,$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, defines a metric on $\mathbb{R}^2$ called the taxicab metric (you do not have to prove that this satisfies the metric axioms).

(a) (2 points) Sketch the ball $B_1(\mathbf{0})$ in this metric, where $\mathbf{0} = (0, 0)$.

(b) (4 points) If $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$ is the usual Euclidean norm in $\mathbb{R}^2$, show that

$$|\mathbf{x} - \mathbf{y}| \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{2}|\mathbf{x} - \mathbf{y}|.$$

Hint. For both inequalities, it may be helpful to square both sides.

(cont.)

(c) (2 points) Show that a sequence $\{\mathbf{x}_n\}$ is Cauchy in $(\mathbb{R}^2, d)$ if and only if it is Cauchy with respect to the standard Euclidean metric.

(d) (2 points) Show that $(\mathbb{R}^2, d)$ is *complete*.

4. (a) (5 points) Let $\{a_n\}$ be a sequence in $[-1, 1]$ with the property that any *convergent subsequence* has limit 0. Show that $\{a_n\}$ itself is then a convergent sequence, and that

$$\lim_{n\to\infty} a_n = 0.$$

Hint. Proceed by contradiction. You will get a point if you can write precisely (in terms of ε etc.) the meaning of the statement that a_n **does not** converge to 0.

(b) (5 points) Let X be a connected metric space which is not bounded. Show that for any $p \in X$, and any $r > 0$, the set

$$S_r := \{x \in X \mid d(p, x) = r\}$$

is non-empty. **Hint.** Again contradiction!