Math 115. A Different Proof of Theorem 7.26

**Theorem.** Let \((x_1, y_1)\) be the smallest positive solution of \(x^2 - dy^2 = 1\). Then the set of all positive solutions is equal to the set

\[
\{ (x_n, y_n) : n \in \mathbb{Z}^+ \},
\]

where \(x_n\) and \(y_n\) are determined by \(x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n\).

**Proof.** First note that if \(x, y \in \mathbb{Z}\) then \((x + y\sqrt{d})(x + y\sqrt{d})' = x^2 - dy^2\), so \((x, y)\) is a solution to \(x^2 - dy^2 = 1\) if and only if \((x + y\sqrt{d})(x + y\sqrt{d})' = 1\).

Next, note that if \((s_1, t_1)\) and \((s_2, t_2)\) are arbitrary solutions to \(x^2 - dy^2 = 1\), and \((s_3, t_3)\) is determined by \((s_1 + t_1\sqrt{d})(s_2 + t_2\sqrt{d}) = s_3 + t_3\sqrt{d}\), then \((s_3, t_3)\) is also a solution. Indeed, for \(i = 1, 2, 3\) let \(\alpha_i = s_i + t_i\sqrt{d}\), so that \(\alpha_3 = \alpha_1\alpha_2\). Then \(\alpha_3\alpha'_3 = (\alpha_1\alpha_2)(\alpha_1\alpha'_2) = (\alpha_1\alpha'_1)(\alpha_2\alpha'_2) = 1 \cdot 1 = 1\), so \((s_3, t_3)\) is a solution.

Moreover, if \((s_1, t_1)\) and \((s_2, t_2)\) are positive solutions, then so is \((s_3, t_3)\), since \(s_3 = s_1s_2 + dt_1t_2 > 0\) and \(t_3 = s_1t_2 + s_2t_1 > 0\).

Now the inclusion \(\supseteq\) follows by repeated application of this fact to get

\[
(x_n + y_n\sqrt{d})(x_n + y_n\sqrt{d})' = 1.
\]

Next, consider the opposite inclusion \(\subseteq\).

First, note that on the curve \(x^2 - dy^2 = 1\) with \(x > 0\) and \(y > 0\), we have:

1. \(x\) is an increasing function of \(y\), because \(x = \sqrt{1+dy^2}\);
2. \(x + y\sqrt{d}\) is an increasing function of \(y\);
3. \(x/y > \sqrt{d}\), because \(x/y = \sqrt{d + 1/y^2}\); and
4. \(x/y < \sqrt{d}\), because \(x/y = \sqrt{d - 1/y^2}\).

Now assume that the inclusion is false, and let \((s, t)\) be the smallest positive solution not of the form \((x_n, y_n)\) for some \(n\). Then \(t > y_1\). Let \(\xi = x_1 + y_1\sqrt{d}\) and \(\alpha = s + t\sqrt{d}\). Since \(\xi\xi' = 1\), we have \(\xi^{-1} = \xi' = x_1 - y_1\sqrt{d}\). Then

\[
\alpha\xi^{-1} = (s + t\sqrt{d})(x_1 - y_1\sqrt{d}) = (sx_1 - dty_1) + (tx_1 - sy_1)\sqrt{d}.
\]

Accordingly, let \(s_2 = sx_1 - dty_1\) and \(t_2 = tx_1 - sy_1\). Then (noting that \((x_1, -y_1)\) is a solution), it follows from the above that \((s_2, t_2)\) is a solution of \(x^2 - dy^2 = 1\). Moreover, it is a positive solution, because

\[
s_2 = ty_1 \left(\frac{x_1}{y_1} - \frac{s}{t} - d\right) > 0\quad\text{and}\quad t_2 = ty_1 \left(\frac{x_1}{y_1} - \frac{s}{t}\right) > 0
\]

by (iv) and (iii), respectively. Finally, since \(\xi > 1\), we have \(0 < \xi^{-1} < 1\), therefore \(\alpha/\xi < \alpha\), and this implies that \(t_2 < t\) by (iii). Then \((s_2, t_2) = (x_n, y_n)\) for some \(n\) by the choice of \((s, t)\). Therefore \((s, t) = (x_{n+1}, y_{n+1})\), contradicting the choice of \((s, t)\). \(\square\)