Math 115. Proof of the Existence of the Greatest-Integer Function

This handout will prove the following lemma mentioned in class:

**Lemma.** For each real number $x$ there is a unique integer $n$ such that

$$n \leq x < n+1.$$  

**Proof.** This proof will rely on the following axiomatic properties.

**Archimedean Property of the Real Numbers.** For each $x \in \mathbb{R}$ there is an integer $n$ with $n > x$.

**Well-Ordering Property of the Natural Numbers.** Every nonempty subset of $\mathbb{N}$ has a smallest element.

Start by letting $x$ be any real number. By the Archimedean Property applied to the real number $-x$, there is an $m \in \mathbb{Z}$ with $m > -x$, so $-m < x$.

Next let

$$S = \{ k \in \mathbb{N} : k - m > x \}.$$  

Note that $0 \notin S$ because $-m < x$ (as noted above).

The set $S$ is a subset of $\mathbb{N}$ by construction, and is nonempty by the archimedean property applied to the real number $x + m$. Therefore, by the well-ordering property, it has a smallest element $k$. Since $k \in S$, $k - m > x$.

Since $k$ is the smallest element of $S$, we have $k - 1 \notin S$. This can happen only if $k - 1 \notin \mathbb{N}$ or $k - 1 - m \leq x$. But $k - 1 \notin \mathbb{N}$ would imply $k < 1$, so $k = 0$, contradicting the fact that $0 \notin S$. Therefore $k - 1 - m \leq x$.

Thus, letting $n = k - 1 - m$, we have

$$n \leq x < n + 1,$$

as was to be shown.  \[\Box\]