Lecture #1

§1. Nevanlinna Theory

Nevanlinna theory is part (most) of value distribution theory of holomorphic functions

Consider the function e^z. It has no zeroes or poles, so as a map $\mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ it omits the values 0 and ∞.

Theorem (Picard). *There is no non-constant holomorphic function $\mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ omitting three or more values.*

[draw $(e^z)^{-1}(2)$]

Note that $(e^z)^{-1}(w) = \begin{cases} \{\log w + 2\pi i n : n \in \mathbb{Z}\} & w \in \mathbb{C} \setminus \{0\}, \\ \emptyset & w = 0, \infty \end{cases}$

$\therefore\ \#\{z \in \mathbb{C} : e^z = w \text{ and } |z| \le r\} = \begin{cases} \frac{r}{\pi} + O_w(1) & w \in \mathbb{C} \setminus \{0\}, \\ 0 & w = 0, \infty \end{cases}$

From now on *assume* $f(0) \ne 0, \infty$.

Let $\log^+ x = \max\{0, \log x\}$. Also let $f \colon \mathbb{C} \to \mathbb{C}$ be meromorphic.

Definition. The **proximity function** is

$$m_f(r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi}, \qquad \text{and}$$

$$m_f(a, r) = m_{1/(f-a)}(r) = -\int_0^{2\pi} \log^- |f(re^{i\theta}) - a| \frac{d\theta}{2\pi} \qquad a \in \mathbb{C}.$$

Also let $m_f(\infty, r) = m_f(r)$.

Definition. The **counting function** is

$$N_f(r) = \sum_{|z|<r} \operatorname{ord}_z^+(1/f) \cdot \log \frac{r}{|z|}, \qquad \text{and}$$

$$N_f(a, r) = N_{1/(f-a)}(r) = \sum_{|z|<r} \operatorname{ord}_z^+(f - a) \cdot \log \frac{r}{|z|}.$$

Also let $N_f(\infty, r) = N_f(r)$.

Finally, we define the **height function** by

$$T_f(r) = m_f(r) + N_f(r).$$

If $f(z) = e^z$ then $N_f(\infty, r) = 0$ and

$$m_f(\infty, r) = \int \log^+ e^{r\cos\theta} \frac{d\theta}{2\pi} = r \int_{-\pi/2}^{\pi/2} \cos\theta \frac{d\theta}{2\pi} = \frac{r}{\pi}.$$

Theorem (First Main Theorem (FMT)). *For all* $a \in \mathbb{C}$,

$$m_f(a,r) + N_f(a,r) = T_f(r) + O_{f,a}(1) \ .$$

Since $m_f(a,r) \ge 0$, this gives an *upper bound* on $N_f(a,r)$.

Compare with **Jensen's formula**

$$\log|c_\lambda| = \int_0^{2\pi} \log|f(re^{i\theta})| \frac{d\theta}{2\pi} + N_f(\infty,r) - N_f(0,r) \ .$$

Theorem (Second Main Theorem (SMT)). *Let* $a_1, \dots, a_q \in \mathbb{P}^1(\mathbb{C})$ *be distinct. Then*

$$(*) \qquad \sum_{i=1}^{q} m_f(a_i,r) \le_{\text{exc}} 2\,T_f(r) + O(\log^+ T_f(r)) + o(\log r) \ ,$$

where $O(\ )$ *and* $o(\ )$ *depend only on* f *and* $a_1, \dots, a_q$, *and* $\le_{\text{exc}}$ *means that the inequality holds for all* $r \in (0,\infty)$ *outside of a set of finite Lebesgue measure.*

Corollary (Picard). *If* $f \colon \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \setminus \{a_1,a_2,a_3\}$ *is holomorphic with* a_1,a_2,a_3 *distinct, then* f *is constant.*

Proof. Since f never equals a_i , we have $N_f(a_i,r) = 0$, so the First Main Theorem gives $m_f(a_i,r) = T_f(r) + O(1)$. The left-hand side of (*) is therefore $3\,T_f(r) + O(1)$, so (*) becomes $T_f(r) \le_{\text{exc}} O(\log^+ T_f(r)) + o(\log r)$. But, if f is nonconstant then $T_f(r) \ge O(\log r)$, a contradiction. Therefore f is constant. $\square$

One can view the SMT as a lower bound on $N_f(a,r)$: the left-hand side of (*) is $q\,T_f(r) - \sum m_f(a_i,r)$, so (*) is equivalent to

$$\sum_{i=1}^{q} N_f(a_i,r) \ge_{\text{exc}} (q-2)T_f(r) + O(\log^+ T_f(r)) + o(\log r) \ .$$

Advantages:

(1). $q - 2 = \chi(\mathbb{P}^1 \setminus q \text{ points})$,
(2). The left-hand side is independent of metrics; and
(3). One can phrase it using truncated counting functions (abc conjecture).

§2. Number Theory

For a number field k, let M_k be its set of places. This is in one-to-one correspondence with the disjoint union

$$\{\text{nonzero primes in } \mathscr{O}_k\} \coprod \{\sigma\colon k \hookrightarrow \mathbb{R}\}$$
$$\coprod \{\text{unordered pairs } (\sigma, \bar\sigma)\colon \sigma \ne \bar\sigma\colon k \hookrightarrow \mathbb{C}\} .$$

For $v \in M_k$ we define *norms* $\|\cdot\|_v$ by

$$\|x\| = \begin{cases} (\mathscr{O}_k : \mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(x)} & \text{if } v \nmid \infty,\ x \ne 0, \\ |\sigma(x)| & \text{if } v \text{ is real,} \\ |\sigma(x)|^2 & \text{if } v \text{ is complex.} \end{cases}$$

We then have a **product formula** $\prod_{v \in M_k} \|x\|_v = 1$ for all $x \in k$, $x \ne 0$.

Let S_∞ denote the set of archimedean (real or complex) places.

Let $S \supseteq S_\infty$ be a finite set of places of k; for $x \in k$ we then define

$$m_S(x) = m_S(\infty, x) = \sum_{v \in S} \log^+ \|x\|_v ,$$

$$m_S(a, x) = m_S\left(\frac{1}{x-a}\right) = \sum_{v \in S} \log^+ \left\|\frac{1}{x-a}\right\|_v ,$$

$$N_S(x) = N_S(\infty, x) = \sum_{v \notin S} \log^+ \|x\|_v = \sum_{v \notin S} \operatorname{ord}_v^+(1/x) \log(\mathscr{O}_k : \mathfrak{p}) ,$$

$$N_S(a, x) = N_S\left(\frac{1}{x-a}\right) = \sum_{v \notin S} \log^+ \left\|\frac{1}{x-a}\right\|_v .$$

$$h_k(x) = m_S(x) + N_S(x) = \sum_{v \in M_k} \log^+ \|x\|_v = \log \prod_v \max\{1, \|x\|_v\} .$$

Corresponding to the FMT, we have

$$m_S(a, x) + N_S(a, x) = h_k\left(\frac{1}{x-a}\right) = h_k(x) + O_{a,k}(1) ,$$

a property of heights.

Theorem (Roth). *Let k and S be as above, and for all $v \in S$ let $\alpha_v \in \overline{\mathbb{Q}}$. Let $\epsilon > 0$.*

Then the inequality

$$\prod_{v \in S} \min\{1, \|x - \alpha_v\|_v\} \le \frac{1}{H_k(x)^{2+\epsilon}} .$$

holds for only finitely many $x \in k$ *.*

This is equivalent to the same statement with $\alpha_v \in k$ for all v (expand k).

Equivalently, given k, S, ϵ, and $a_1, \dots, a_q \in k$, then the inequality

$$\prod_{i=1}^{q} \prod_{v \in S} \min\{1, \|x - a_i\|_v\} \le \frac{1}{H_k(x)^{2+\epsilon}}$$

holds for only finitely many $x \in k$.

Taking $-\log$ of both sides, and rearranging the logic, we then have that

$$\sum_{i=1}^{q} m_S(a_i, x) \le (2+\epsilon) h_k(x) + O(1)$$

for almost all $x \in k$.

§3. Schmidt's Subspace Theorem and a Theorem of Cartan

We start with Nevanlinna theory. Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ ($n > 0$), defined by $a_0x_0 + \cdots + a_nx_n = 0$. For $P \in \mathbb{P}^n(\mathbb{C})$, $P \notin H$, with homogeneous coordinates $[z_0 : \cdots : z_n]$, let

$$\lambda_H(P) = -\frac{1}{2}\log\frac{|a_0z_0 + \cdots + a_nz_n|^2}{|z_0|^2 + \cdots + |z_n|^2}\ ,$$

and for a holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with image not contained in H, define the proximity function

$$m_f(H,r) = \int_0^{2\pi} \lambda_H(f(re^{i\theta}))\frac{d\theta}{2\pi}\ .$$

For the counting function, note that f^*H is an analytic divisor on $\mathbb{C}$; we can then define

$$N_f(H,r) = \sum_{|z|<r} \operatorname{ord}_z f^*H \cdot \log\frac{r}{|z|}\ .$$

For the height function, we note that for any hyperplane H not containing the image of f,

$$T_f(r) := m_f(H,r) + N_f(H,r)$$

is independent of H up to $O(1)$, with an implicit constant depending only on H.

Theorem (Cartan). *Let $n > 0$ and let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position (i.e., every intersection of $r \le n$ of them has codimension r). Let $f\colon \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map whose image is not contained in any hyperplane. Then*

$$\sum_{i=1}^q m_f(H_i,r) \le_{\text{exc}} (n+1)T_f(r) + O(\log^+ T_f(r)) + o(\log r)\ .$$

In number theory, things are very similar. Let k and S be as above, and let H be a hyperplane in $\mathbb{P}^n_k$ given by $a_0x_0 + \cdots + a_nx_n = 0$ (with $a_i \in k$ for all i). For $P \in P^n(k) \setminus H$ with homogeneous coordinates $[x_0 : \cdots : x_n]$ (again with $x_i \in k$ for all i), and $v \in M_k$ we define

$$\lambda_{H,v}(P) = -\log\frac{\|a_0x_0 + \cdots + a_nx_n\|_v}{\max\{\|x_0\|_v, \dots, \|x_n\|_v\}}\ ,$$

and define a proximity function $m_S(H,P) = \sum_{v\in S}\lambda_{H,v}(P)$, a counting function $N_S(H,P) = \sum_{v\notin S}\lambda_{H,v}(P)$, and a height function

$$\begin{aligned} h_k(P) = m_S(H,P) + N_S(H,P) &= \sum_{v\in M_k} -\log\frac{\|a_0x_0 + \cdots + a_nx_n\|_v}{\max\{\|x_0\|_v, \dots, \|x_n\|_v\}} \\ &= \sum_v \log\max\|x_i\|_v \end{aligned}$$

by the product formula.

Theorem (Schmidt's Subspace Theorem). *Let* k , S , $\epsilon > 0$, *and* $n > 0$ *be as above, and let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n_k$ *in general position. Then*

$$\sum_{i=1}^{q} m_S(H_i, x) \le (n+1+\epsilon) h_k(x) + O(1)$$

for all $x \in \mathbb{P}^n(k)$ *outside of a finite union of proper linear subspaces of* $\mathbb{P}^n_k$.

Actually, Schmidt had a collection $H_{v,0}, \dots, H_{v,n}$ for each $v \in S$, in general position for each v . The union, however, did not need to be in general position. This is equivalent to:

Theorem (Schmidt's Subspace Theorem). *Let* k , S , $\epsilon > 0$, *and* $n > 0$ *be as above, and let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n_k$. *Then the inequality*

$$\sum_{v \in S} \max_J \sum_{i \in J} \lambda_{H_i, v}(x) \le (n+1+\epsilon) h_k(x) + O(1)$$

holds for all $x \in \mathbb{P}^n(k)$ *outside of a finite union of proper linear subspaces. Here* J *varies over a given collection of subsets of* $\{1, \dots, q\}$ *for which* $\{H_i : i \in J\}$ *lie in general position.*

Theorem (Cartan). *Let* $H_1, \dots, H_q$ *be hyperplanes in* $\mathbb{P}^n(\mathbb{C})$ *and let* $f \colon \mathbb{C} \to \mathbb{P}^n$ *be a holomorphic curve whose image is not contained in any hyperplane. Then*

$$\int_0^{2\pi} \max_J \sum_{i \in J} \lambda_{H_i}(f(re^{\sqrt{-1}\theta})) \frac{d\theta}{2\pi} \le_{\text{exc}} (n+1) T_f(r) + O(\log^+ T_f(r)) + o(\log r)$$

(where J *is as before).*

§4. The Dictionary

- One holomorphic map corresponds to an *infinite* set of rational points.
- One rational point may correspond to $f\big|_{\bar{\mathbb{D}}_r}$.
- $r \in (0,\infty)$ corresponds to $x \in k$
- $\theta \in [0,2\pi]$ corresponds to $v \in S$
- $|f(re^{i\theta})|$ corresponds to $\|x\|_v$, $v \in S$
- $\operatorname{ord}_z f$ ($|z| < r$) corresponds to $\operatorname{ord}_v x$, $v \notin S$
- $\log \dfrac{r}{|z|}$ corresponds to $\log(\mathscr{O}_k : \mathfrak{p})$

[draw $\mathbb{D}_r$]

§5. The Borel Lemma/Unit Theorem

Lemma (Borel). *If $g_1, \dots, g_n$ are entire functions such that*

$$e^{g_1} + \dots + e^{g_n} = 1$$

then some g_i is constant.

Proof. $[e^{g_1} : \dots : e^{g_n}]$ gives a holomorphic map $f\colon \mathbb{C} \to \mathbb{P}^{n-1}$ missing the n coordinate hyperplanes and also the hyperplane $x_0 + \dots + x_n = 0$. Therefore

$$m(H_i, r) = T_f(r) + O(1)$$

for each of these $n+1$ hyperplanes, so f must be linearly degenerate. We can then eliminate one of the g_i and then use induction on n. □

Lemma (Schlickewei, van der Poorten). *Given a collection of n-tuples $(u_1, \dots, u_n)$ of units in $\mathscr{O}_k$ satisfying $u_1 + \dots + u_n = 1$, all but finitely many have the property that there is a proper subset I of $\{1, \dots, n\}$ with at least two elements and having the property that $\sum_{i \in I} u_i = 0$.*

Proof. $N_S(H_i, [u_1 : \dots : u_n]) = 0$ for the same hyperplanes as before, so the given points all lie in a finite union of proper linear subspaces. Therefore there is an infinite subset lying in a hyperplane. □

Why do these theorems correspond?

Proofs of results relying on these two theorems also correspond.

§6. Weil Functions

There is a unique way (up to a rather involved definition of $O(1)$) to assign to each pair (X, D), where X is a complete k-variety and D is a Cartier divisor on X, a Weil function

$$\lambda_D \colon \coprod_{v \in M_k} (X \setminus D)(\bar{k}_v) \to \mathbb{R} ,$$

such that

(i). normalization: if $H \subseteq \mathbb{P}^n$ is the hyperplane at infinity, then

$$\lambda_{H,v}([x_0 : \cdots : x_n]) = -\log \frac{\|x_0\|_v}{\max \|x_i\|_v} + O(1) ,$$

(ii). additivity: $\lambda_{D+D'} = \lambda_D + \lambda_{D'} + O(1)$, and

(iii). functoriality: if $f \colon X \to Y$ is a morphism and D is a divisor on Y whose support does not contain the image of X, then $\lambda_{f^*D}(x) = \lambda_D(f(x)) + O(1)$.

(iv). continuity: $\lambda_{D,v}$ is continuous in the v-topology.

Here $\lambda_{D,v}$ means the restriction of λ_D to $(X \setminus D)(\bar{k}_v)$, and additivity is only assumed to hold on the intersection of the domains.

We also have $h_{\mathscr{O}(D),k}(x) = \sum_{v \in M_k} \lambda_{D,v}(x) + O(1)$ for all $x \notin \operatorname{Supp} D$. This also works for points over $\bar{k}$, with suitable adjustments.

We can then define a proximity function
$m_S(D, P) = \sum_{v \in S} \lambda_{D,v}(P)$ and a counting function
$N_S(D, P) = \sum_{v \notin S} \lambda_{D,v}(P)$ for all $P \in (X \setminus D)(k)$.

Again, $m_S(D, P) + N_S(D, P) = h_{\mathscr{O}(D),k}(P) + O(1)$.

In Nevanlinna theory itself: Let X be a complete complex variety and let D be a Cartier divisor on X. Choose a hermitian metric on $\mathscr{O}(D)$, and let 1_D denote the canonical section. Then $\lambda_D := -\frac{1}{2} \log |1_D|^2$ gives a Weil function (which is a much simpler notion in this context).

For a holomorphic map $f \colon \mathbb{C} \to X$ whose image is not contained in the support of D, we may then define a proximity function

$$m_f(D, r) = \int_0^{2\pi} \lambda_D(f(re^{i\theta})) \frac{d\theta}{2\pi}$$

and counting function $N_f(D, r) = \sum_{|z|<r} \operatorname{ord}_z f^*D \cdot \log \frac{r}{|z|}$.

§7. Integral Points

Let k and S be as above, let X be a complete variety over k, and let D be an effective divisor on X such that $X \setminus \operatorname{Supp} D$ is affine. Naïvely, a point $P \in X(k)$ is a (D, S)**-integral point** on X if there is a rational map $i\colon X \dashrightarrow \mathbb{P}^n$ for some n, defined and injective on $X \setminus \operatorname{Supp} D$, such that $\operatorname{Supp} D = i^{-1}(\{x_0 = 0\})$ and $i(P) \in \mathbb{A}^n(\mathscr{O}_{k,S})$.

However, this notion is useful only for infinite sets (for any finite set you can clear the denominators) (Serre). This definition can be expressed using Weil functions:

Definition. A subset Σ of $X(k)$ is (D, S)**-integral** if there is a Weil function λ_D for D and constants $c_v \in \mathbb{R}$ for all $v \notin S$ such that $c_v = 0$ for almost all v and $\lambda_{D,v}(P) \le c_v$ for all $v \notin S$ and all $P \in \Sigma$.

If $D = i^*(\{x_0 = 0\})$, then this is easily seen to be equivalent to the earlier naive definition if i is an actual morphism, by functoriality and normalization of Weil functions. Moreover, by additivity of Weil functions this notion depends only on $\operatorname{Supp} D$. Indeed, if D and D' are effective divisors with the same support, then $D \le mD'$ for some m, and so $\lambda_D \le m\lambda_{D'} + O(1)$. It also depends only on the quasi-projective variety $X \setminus \operatorname{Supp} D$.

This definition does not assume that $X \setminus D$ is affine. This is useful, e.g., for $\mathscr{A}_{g,n}$ (the moduli space of principally polarized abelian varieties of given dimension and level structure). Also, if $D = 0$ then the condition is vacuous and all sets of rational points are integral.

Also, note that this definition is well-behaved with respect to morphisms of $X \setminus D$: given morphisms $\phi\colon X \to X'$ and effective divisors D on X and D' on X' such that $\phi(X \setminus \operatorname{Supp} D) \subseteq X' \setminus \operatorname{Supp} D'$, then $\operatorname{Supp} D \supseteq \phi^{-1}(\operatorname{Supp} D')$, so $\phi^* D' \le nD$. Therefore, by functoriality and linearity of Weil functions, (D, S)-integral points on X are mapped to (D', S)-integral points on X'.

Of course, (D, S)-integrality implies that $N_S(D, P)$ is bounded, and this corresponds to $N_f(D, r)$ being bounded in Nevanlinna theory. One important case in which this holds is when f misses the support of D, and one notes of course that maps to $X \setminus \operatorname{Supp} D$ are also well-behaved with respect to morphisms $\phi\colon X \to X'$ taking $X \setminus D$ to $X' \setminus D'$.

Lecture #2

§1. Conjectures

Conjecture. *Let k be a number field, let $S \supseteq S_\infty$ be a finite set of places of k, let X be a smooth projective variety over k, let D be a normal crossings divisor on X, let A be an ample divisor on X, let $\mathscr{K}$ denote the canonical line sheaf on X, and let $\epsilon > 0$. Then there is a proper Zariski-closed subset Z of X, depending only on the above data, such that*

$$m_S(D,x) + h_{\mathscr{K},k}(x) \le \epsilon\, h_{A,k}(x) + O(1)$$

for all $x \in (X \setminus Z)(k)$.

Conjecture. *Let X be a smooth projective complex variety, let D be a normal crossings divisor on X, let A be an ample divisor on X, and let $\mathscr{K}$ denote the canonical line sheaf on X. Then there is a proper Zariski-closed subset Z of X, depending only on the above data, such that for all holomorphic maps $f\colon \mathbb{C} \to X$ with $f(\mathbb{C}) \nsubseteq Z$,*

$$m_f(D,r) + T_{\mathscr{K},f}(r) \le_{\text{exc}} O(\log^+ T_{A,f}(r)) + o(\log r)\ .$$

These are proved if $\dim X = 1$, or if $X = \mathbb{P}^n$ and D is a sum of hyperplanes. Limited results have also been shown if X is a certain type of compactification of a semiabelian variety, or if X is a completion of a Shimura variety.

In other words, we know very little if $\dim X > 1$.

§2. Embeddings

For example, consider $\mathbb{P}^2$, and let D be as above. If $\deg D \ge 4$ then $\mathbb{P}^2 \setminus D$ should not have a dense set of integral points (or admit a Zariski-dense holomorphic curve to it).

If D is smooth of degree ≥ 4, then there's no hope (so far). For four lines, however, we do have results (Schmidt and Cartan).

For three lines and a conic, there are some results; e.g., if L_1, L_2, L_3 are linear forms defining the lines and Q is a quadratic polynomial defining the conic, then all L_i^2/Q must be units (or nearly so) at integral points, so we can apply the unit lemma (or Borel's lemma) (M. Green and V.)

Besides that, how do we get *any* results for a conic?

Under the 2-uple embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, the image of a conic is contained in a hyperplane. Under the 3-uple embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$, the image of a conic spans a linear subspace of codimension 3, so we can get 3 linearly independent hyperplanes containing it.

More generally, say you have a divisor D of degree d in $\mathbb{P}^n$, and look at its image under the r-uple embedding

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{r+n}{n}-1} .$$

The image of the divisor spans a linear subspace of codimension $\binom{r+n-d}{n}$, because there are that many monomials of degree $n-d$ (which then get multiplied by the form defining D to get homogeneous polynomials of degree n in the original variables, hence hyperplanes in the image space).

So the inequality we'd get out of applying Schmidt's Subspace Theorem to $\mathbb{P}^{\binom{r+n}{n}-1}$ would look like

$$\binom{r+n-d}{n} m(D,x) + \dots \le \left(\binom{r+n}{n}\cdot r + \epsilon\right) h_k(x) + O(1)$$

for $x \in \mathbb{P}^n(k)$ outside of a finite union of proper subvarieties of degree $\le r$.

So, as $r \to \infty$, we're interested in the ratio

$$\frac{\binom{r+n-d}{n}}{\binom{r+n}{n}\cdot r} = \frac{(r-d+n)\cdots(r-d+1)}{(r+n)\cdots(r+1)r} = \frac{r^n + O(r^{n-1})}{r(r^n + O(r^{n-1}))} \to \frac{1}{r} \to 0 .$$

This is not very promising, but we can try harder. We can also consider that some hyperplanes can be made to contain D twice, or three times, etc. This gives something like

$$\begin{aligned} &\left(\binom{r+n-d}{n} + \binom{r+n-2d}{n} + \dots\right) m(D,x) + \dots \\ &\quad \le \left(\binom{r+n}{n}\cdot r + \epsilon\right) h_k(x) + O(1) \end{aligned}$$

To estimate the factor in front of $m(D,x)$:

$$\binom{r-kd+n}{n} = \frac{(r-kd+n)\cdots(r-kd+1)}{n!} = \frac{(r-kd)^n + O_n((r-kd)^{n-1})}{n!}$$

and therefore

$$\sum_{k=1}^{[r/d]} \binom{r-kd+n}{n} = \frac{(r-d)^n + \cdots + (r-[r/d]d)^n + O_{n,d}(r^n)}{n!} .$$

As $r \to \infty$, the ratio of interest now converges to

$$\frac{\sum_{k=1}^{[r/d]}\binom{r-kd+n}{n}}{r\binom{r+n}{n}} \approx \frac{\frac{r^{n+1}}{(n+1)d\cdot n!}}{\frac{r^{n+1}}{n!}} \to \frac{1}{(n+1)d} .$$

This *would* give us something like

$$d\, m_S(D,x) + \cdots \le (n+1+\epsilon)h_k(x) + O(1) .$$

This is best possible if $d=1$, but is still new and noteworthy if $d>1$.

But there is something wrong with this.

Suppose we have two divisors, and their images under the r-uple embedding span linear subspaces L_1 and L_2 of codimensions ρ_1 and ρ_2, respectively. We have

$$\operatorname{codim}(L_1 \cap L_2) \le \rho_1 + \rho_2$$

(assuming $L_1 \cap L_2 \ne \emptyset$). If this inequality is strict then we have problems to the extent that $m(L_1,y)$ and $m(L_2,y)$ come from $m(L_1\cap L_2, y)$ (where y is the image of x under the r-uple embedding). Say you have ρ_1 generic hyperplanes containing L_1 and ρ_2 generic hyperplanes containing L_2. If these $\rho_1+\rho_2$ hyperplanes are in general position then this implies

$$\operatorname{codim}(L_1 \cap L_2) \ge \rho_1 + \rho_2 .$$

So we need equality. However, the usual computation of Hilbert function using short exact sequences gives (for sufficiently large n) that the codimension of the linear span of $D_1 \cap D_2$ is

$$\binom{n-d_1+r}{n} + \binom{n-d_2+r}{n} - \binom{n-d_1-d_2+r}{n} ,$$

which is too small by $\binom{n-d_1-d_2+r}{n}$.

[Lecture continued on the blackboard.]

Lecture #4

§1. More abc

One can also get the abc conjecture from the earlier conjecture on rational points (for varieties of large dimension).

For n large let X_n be the variety in $(\mathbb{P}^2)^n$ in coordinates

$$([x_1 : y_1 : z_1], \dots, [x_n : y_n : z_n])$$

given by the equation $\prod x_i^i + \prod y_i^i + \prod z_i^i = 0$, and let Γ_n be the closure of the graph of the rational map $X_n \dashrightarrow \mathbb{P}^2$ given by

$$([x_1 : y_1 : z_1], \dots, [x_n : y_n : z_n]) \mapsto \left[\prod x_i^i : \prod y_i^i : \prod z_i^i\right] .$$

Let $\phi\colon \Gamma_n \to \mathbb{P}^2$ be the resulting morphism. The image of ϕ is a line in $\mathbb{P}^2$, which we identify with $\mathbb{P}^1$.

Given relatively prime $a, b, c \in \mathbb{Z}$ with $a + b + c = 0$, we get a point $P_{a,b,c}$ on Γ_n by letting

$$x_n = \prod_p p^{[\operatorname{ord}_p a/n]} , \qquad x_i = \prod_{\operatorname{ord}_p a \equiv i \pmod n} p \quad (i < n) ,$$

so that $a = \prod x_i^i$ with x_n as large as possible. Similarly define the y_i using b and the z_i using c. Let D be the divisor $x_1 \cdots x_n y_1 \cdots y_n z_1 \cdots z_n = 0$ on Γ_n. It is then possible to show that

$$N(D, P_{a,b,c}) \le \sum_{p \mid abc} \log p + \frac{1}{n} N(E, \phi(P_{a,b,c})) ,$$

where E is the divisor $[0] + [1] + [\infty]$ on the image $\mathbb{P}^1$ of ϕ.

There are additional obstacles to overcome, but basically the conjecture for rational points applied to a certain desingularization of Γ_n then gives the abc conjecture.

We also note that the group $\mathbb{G}_{\mathrm{m}}^{2n-2}$ acts on X_n, and therefore on Γ_n, by $x_i \mapsto t x_i$, $x_n \mapsto t^{-i} x_n$ ($1 \le i < n$) for the first $n - 1$ factors, and similarly with the y_i for the remaining factors. This action preserves the divisor D and is faithful.

This implication therefore relies only on the conjecture for rational points for varieties X and divisors D for which there is a semiabelian variety G of dimension $\dim X - 1$ acting faithfully on X in such a way as to preserve D. In this special case, the conjecture has been proved in the split function field case and in the Nevanlinna case.

It's actually pretty easy to end up in "abc land."

§2. The $1+\epsilon$ conjecture

Recall that the $1+\epsilon$ conjecture states that if X is a smooth projective curve over a number field k then for all algebraic points x of bounded degree,

$$h_K(x) \le (1+\epsilon)d_k(x) + O(1) .$$

We can prove this in the "split" function field case, as follows (essentially following de Franchis):

Let C be a smooth projective curve over a ground field F of characteristic 0, and let $k = K(C)$. Let X_0 be a smooth projective curve over F and D_0 a reduced effective divisor on X_0. Then $X := X_0 \times_F C \to C$ is a model for $X_0 \times_F k$, and D_0 extends to $\operatorname{pr}_1^* D_0$ on X, where pr_1 is the projection of $X = X_0 \times C$ to its first factor.

An algebraic point x on X_k corresponds to a nonsingular finite cover E of C of degree d and a generically injective C-morphism $i\colon E \to X$. We have

$$d_k(x) = (2g(E)-2)/d - (2g(C)-2) ,$$

which is the degree of the ramification divisor divided by d. Also $K_{X/C} = \operatorname{pr}_1^* K_{X_0}$, and the height of x relative to this canonical divisor is

$$\frac{(2g(X_0)-2)\deg(\operatorname{pr}_1 \circ i)}{d} \le \frac{2g(E)-2}{d} = d_k(x) + (2g(C)-2) . \qquad \square$$

A way to look at this proof is that we know the derivative of E, as an element of the absolute tangent bundle $T_{X/F}$. Since we're in the "split" case, $T_{X/F}$ splits into a product $T_{X/X_0} \times T_{X/C}$, and we can project onto the second factor, the *relative* tangent bundle.

As I see it, a key aspect of McQuillan's proof of the $1+\epsilon$ conjecture in the (general) function field case of characteristic zero is the observation that although you don't have a canonical projection of the absolute tangent bundle to the relative tangent bundle anymore, you can choose a projection arbitrarily, and for points of large height the value of the projection doesn't vary that much with the projection, in proportion with the tangent vector.

§3. Derivatives

One of the key tools in Nevanlinna theory is the following:

Theorem (Lemma on the Logarithmic Derivative (LLD)) (Nevanlinna). *Let f be a meromorphic function on $\mathbb{C}$. Then*

$$\int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \frac{d\theta}{2\pi} \le_{\text{exc}} O(\log^+ T_f(r)) + o(\log r) .$$

Proof. Omitted (not instructive).

Theorem (Geometric LLD (GLLD) (Wong, McQuillan)). *Let X be a smooth projective complex variety, let D be a normal crossings divisor on X, and let $f\colon \mathbb{C} \to X$ be a holomorphic curve with image not contained in* $\operatorname{Supp} D$*. Let $\mathscr{A}$ be an ample line sheaf on X. Let $|\cdot|$ be a hermitian metric on the log tangent bundle $T_X(-\log D)$, and let $d_D f\colon \mathbb{C} \to T_X(-\log D)$ denote the canonical lifting of f. Then*

$$\int_0^{2\pi} \log^+ \left| d_D f(re^{i\theta}) \right| \frac{d\theta}{2\pi} \le_{\text{exc}} O(\log^+ T_{\mathscr{A},f}(r)) + o(\log r) .$$

Proof. Again omitted. Basically it uses the classical LLD locally and applies compactness.

Remark. The case $X = \mathbb{P}^1$, $D = \{0, \infty\}$ recovers the earlier LLD.

This in turn implies:

Theorem (McQuillan's "tautological inequality"). *Let X, D, $f\colon \mathbb{C} \to X$, and $\mathscr{A}$ be as above. Let $f'\colon \mathbb{C} \to \mathbb{P}(\Omega^1_X(\log D))$ be the canonical lifting of f, and let $\mathscr{O}(1)$ denote the tautological line sheaf on $\mathbb{P}(\Omega^1_X(\log D))$. (Here $\mathbb{P}()$ is as in EGA.) Then*

$$T_{\mathscr{O}(1),f'}(r) \le_{\text{exc}} N^{(1)}_f(D,r) + O(\log^+ T_{\mathscr{A},f}(r)) + o(\log r) .$$

Proof (sketch). Let $V = \mathbb{V}(\Omega^1_X(\log D))$, the total space of $T_X(-\log D)$, let

$$\overline{V} = \mathbb{P}(\Omega^1_X(\log D) \oplus \mathscr{O}_X) ,$$

the projective closure of V, let $[\infty] = \overline{V} \setminus V$, a divisor on $\overline{V}$, let $p\colon P \to \overline{V}$ be the blowing-up of $\overline{V}$ along the zero section of V, let $[0] \supseteq P$ be the exceptional divisor, and let $\phi\colon \mathbb{C} \to P$ be the lifting of $d_D f\colon \mathbb{C} \to V$.

We have a diagram ... of X-schemes and $d_D f = p \circ \phi$, $f' = q \circ \phi$. In addition,

$$q^*\mathscr{O}(1) \cong p^*\mathscr{O}(1) \otimes \mathscr{O}(-[0]) .$$

Also (on $\overline{V}$) $\mathscr{O}(1) \cong \mathscr{O}([\infty])$, since $[\infty]$ is cut out by the global section $(0,1)$ of $\Omega^1_X(\log D) \oplus \mathscr{O}_X$.

Therefore

$$\begin{aligned} T_{\mathscr{O}(1),f'}(r) &= T_{q^*\mathscr{O}(1),\phi}(r) = T_{\mathscr{O}(1),d_D f}(r) - T_{\mathscr{O}([0]),\phi}(r) \\ &= N_{d_D f}([\infty],r) + m_{d_D f}([\infty],r) - N_\phi([0],r) - m_\phi([0],r) \\ &\le_{\text{exc}} N^{(1)}_f(D,r) + O(\log^+ T_{\mathscr{A},f}(r) + o(\log r) - 0 - 0 \;. \end{aligned}$$

Explanation: The first term $\le N^{(1)}_D$ because $d_D f$ is bounded except where f hits D, and at that point it has a simple pole (df/f has a simple pole at worst).

The second term is bounded by the GLLD.

The third and fourth terms are ≥ 0, and can be ignored. □

Applications:

(1) The classical SMT for curves:

$$N^{(1)}_f(D,r) \ge_{\text{exc}} T_{K+D,f}(r) + S()$$

Proof. $\Omega^1_X(\log D) = \mathscr{O}(K_X + D)$ and $\mathbb{P}(\Omega^1_X(\log D)) = X$, with $\mathscr{O}(1) = \mathscr{O}(K_X + D)$. Therefore we get the inequality directly. □

(2) Cartan's theorem (using a modified version with varying D)

(3) Results on holomorphic curves in closed subvarieties of semiabelian varieties (hoped for, but not yet proved).

(4) The same for open subvarieties of semiabelian varieties (again, hoped for but not yet proved).

(5) Shimura varieties (again, not yet).

Why is this interesting to number theorists? To push the analogy further, we need to know what is the analogue of the derivative of a holomorphic function.

Comparing Schmidt's proof with a proof of Cartan's theorem by H. Weyl, J. Weyl, and L. Ahlfors suggests that the analogue should involve Minkowski's theory of successive minima, applied on the relative (co)tangent bundle.

Phrased in Arakelov theory, successive minima reduces to line subbundles of maximal degree, so we want a line subbundle of $T_{X/Y}(-\log D)$ of largest Arakelov degree, and therefore a quotient subbundle of $\Omega_{X/Y}(\log D)$ of small degree. This corresponds to a point in $\mathbb{P}(\Omega_{X/Y}(\log D))$ lying over $P \in X(k)$; i.e., a Y-section in $\mathbb{P}(\Omega_{X/Y}(\log D))$ lying over a section of $X \to Y$ for which the corresponding quotient $\Omega_{X/Y}(\log D) \twoheadrightarrow \mathscr{L}$ has small Arakelov degree. What is $\mathscr{L}$? It is the restriction of $\mathscr{O}(1)$, so its degree is the height of this point relative to $\mathscr{O}(1)$.

This leads to...

Conjecture ("Tautological conjecture"). *Let* k , S , X , D , *and* ϵ *be as usual. For all* $P \in X(k)$ *(or* $X(\bar{k})$ *with* $[k(P) : k]$ *bounded) there is a* $P' \in \mathbb{P}(\Omega_{X/Y}(\log D))$ *lying over* P *with*

$$h_{\mathscr{O}(1),k}(P') \le N_S^{(1)}(D,P) + \epsilon h_{A,k}(P) + O(1) \ .$$

This makes sense only in the context of infinite sequences, and preferably:

Definition (Zhang). A **generic sequence** on X is an infinite sequence of points in $X(\bar{k})$ such that all infinite subsequences are Zariski dense.

Proposition (Arithmetic chain rule). *Let* $f\colon X_1 \to X_2$ *be a morphism of* k*-varieties. Then, for all* $P \in X(\bar{k})$ *where* f *is étale, and for all* $P' \in \mathbb{P}(\Omega_{X/Y})$ *lying over* P *, the rational map* $f_*\colon \mathbb{P}(\Omega_{X_1/Y}) \dashrightarrow \mathbb{P}(\Omega_{X_2/Y})$ *takes* P' *to* P'' *(lying over* $f(P)$ *) for which*

$$h_{\mathscr{O}(1),k}(P'') \le h_{\mathscr{O}(1),k}(P') + O(1)$$

where the constant in $O(1)$ *depends only on* f *.*

Proof. Extend X_1 and X_2 to models $\mathscr{X}_1$ and $\mathscr{X}_2$ over $\mathscr{O}_k$ for which f extends as a morphism, and let $i\colon E \to \mathscr{X}_1$ be the section corresponding to P . Then P' corresponds to a surjection $i^*\Omega_{X_1/\mathscr{O}_k} \twoheadrightarrow \mathscr{L}$, and $h_{\mathscr{O}(1),k}(P')$ is the Arakelov degree of $\mathscr{L}$ divided by $[K(E) : k]$. We also have a morphism $f^*\Omega_{X_2/\mathscr{O}_k} \to \Omega_{X_1/\mathscr{O}_k}$, isomorphic at P' . Therefore we get a nonzero map $(f \circ i)^*\Omega_{X_2/\mathscr{O}_k} \to \mathscr{L}$, so $h_{\mathscr{O}(1),k}(P'') \le h_{\mathscr{O}(1),k}(P')$ (with heights defined using these models). ☐

A similar result holds for closed immersions (without the assumption on étaleness).

Both Schmidt's proof of his Subspace Theorem and Ahlfors' proof of Cartan's theorem have two parts—an old part and a new part. The old parts look like the case for dimension 1 , with some necessary modifications. The new parts both involve multilinear algebra, and derivatives of the curve in Ahlfors' case, or successive minima in Schmidt's case. Similarities in the latter parts is what suggested the "tautological conjecture" (as well as McQuillan's result in Nevanlinna theory).

I've been trying for some time to get these two proofs to line up better, but Ahlfors' proof considered the hyperplanes separately when going up to higher exterior powers, whereas Schmidt considered them together. In 2006 I got a variant of Ahlfors' proof that was more like Schmidt's in this respect.

This used a modified tautological inequality, with varying D (similar to what was done with Cartan's theorem).

Define a function μ_D on X measuring the improvement when you add D to McQuillan's inequality, so that the inequality with D is roughly comparable to

$$T_{\mathscr{O}(1),f'}(r) + \int_0^{2\pi} \mu_D(f(re^{i\theta}))\frac{d\theta}{2\pi} \le_{\text{exc}} S() \ .$$

Then the modified McQuillan inequality is

$$T_{\mathscr{O}(1),f'}(r) + \int_0^{2\pi} \max_j \mu_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \le_{\mathrm{exc}} S()\ .$$

My proof of Cartan's theorem used that as the *only* analysis.