Math 256B. Homework #10
Due May 14, online

1. Let \( X \) be a projective scheme over a field \( k \) with \( \dim X > 0 \), and let \( \mathcal{L} \) be a very ample line sheaf on \( X \) (over \( k \)). Show that there exists an integer \( m_0 \) such that the map
\[
H^0(X, \mathcal{L}^\otimes m)^\otimes n \to H^0(X, \mathcal{L}^\otimes mn)
\]
is surjective for all \( m \geq m_0 \) and all \( n \in \mathbb{N} \). (Use the convention that if \( V \) is a vector space over \( k \), then \( V^\otimes 0 = k \).)

2(nc). Let \( k \) be a field. Find the dualizing sheaf of \( V(xy) \) in \( \mathbb{P}^2_k \) (the union of two lines in \( \mathbb{P}^2_k \) intersecting in a point, with reduced induced subscheme structure).

[Hint: Don’t work too hard.]

3. Hartshorne III Ex. 10.1. You may assume that \( p \neq 2 \).

4. (Based on Hartshorne III Ex. 10.6.) Let \( k \) be a field of characteristic \( \neq 2, 3 \), let \( Y \) be the nodal curve \( y^2 = x^2(x+1) \) in \( \mathbb{A}^2_k \), let \( X = \text{Spec} \, k[z,w]/(w^2 - (z^2 - 1)^2) \), and let \( f: X \to Y \) be the map \( (z,w) \mapsto (z^2 - 1, wz) \).

Show that \( f \) is a finite étale morphism of degree 2, and that \( X \) is the union of two irreducible components, each one isomorphic to the normalization of \( Y \) (Fig. 12 in the book).

If you prefer, you may assume that \( k \) is algebraically closed.

5(nc). Let \( k \) be an algebraically closed field, let \( X \) and \( Y \) be schemes of finite type over \( k \), and let \( f: X \to Y \) be a smooth morphism of relative dimension \( n \). Recall that the relative tangent space of \( X/Y \) at a point \( x \in X \) is the dual \( (\Omega_{X/Y} \otimes k(x))^\vee \).

Show that for any closed point \( x \in X \) and any linear subspace \( L \subseteq (\Omega_{X/Y} \otimes k(x))^\vee \), there exist an open neighborhood \( U \) of \( x \) in \( X \) and a closed subscheme \( Z \) of \( U \) passing through \( x \), which is smooth over \( Y \), and whose relative tangent space at \( x \) is \( L: (\Omega_{Z/Y} \otimes k(x))^\vee = L \).