Math 256B. Spectral Sequences

For concreteness, we work over the category $\mathfrak{Ab}$ of abelian groups; however, everything will still work over an arbitrary abelian category.

Mostly this follows Lang's *Algebra,* and also Vakil's *FOAG.* (Note that Vakil interchanges the roles of p and q.)

First, we give the definition of spectral sequence that is likely the most familiar to mathematicians in algebraic geometry.

Definition. A **spectral sequence** is a sequence $\{E_r, d_r\}_{r\ge 0}$ of bigraded objects

$$E_r = \bigoplus_{p,q\in\mathbb{N}} E_r^{p,q} ,$$

together with homomorphisms (called **differentials**) $d_r = d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ (hence of bidegree $(r, 1-r)$) for all r,p,q, such that

(i). $d_r^2 = 0$, and

(ii). $H(E_r) = E_{r+1}$ (i.e.,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}) / \operatorname{im}(d_r^{p-r,q+r-1} \colon E_r^{p-r,q-r+1} \to E_r^{p,q})$$

for all r,p,q).

In the above, we also let $E_r^{(p,q)} = 0$ for all $r \in \mathbb{N}$ and all $(p,q) \in \mathbb{Z}^2 \setminus \mathbb{N}^2$.

Remark. Let $p,q\in\mathbb{N}$ and $n = p+q$. Then, for all $r > n+1$, we have $q-r+1 < 0$ and $p - r < 0$ (since $p,q \le n$); hence $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$, and consequently

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \dots .$$

We let $E_\infty^{p,q}$ denote this limiting value.

Definition. Let $(K^\cdot, D)$ be a (co)complex of abelian groups. Then a **filtration** of $(K^\cdot, D)$ is an $\mathbb{N}$-graded filtration $K^n = F^0K^n \supseteq F^1K^n \supseteq \dots$ of K^n for all $n\in\mathbb{N}$ such that $D(F^pK^n) \subseteq F^pK^{n+1}$ for all n,p. We also assume that $F^pK^n = 0$ for all sufficiently large p, depending on n.

Definition. A **filtered complex** is a complex $(K^\cdot, D)$ with a filtration.

Definition. Let $(K^\cdot, D)$ be a filtered complex. Then, for all $n \in \mathbb{N}$, we define a filtration $\{F^pH^n(K^\cdot)\}_{p\in\mathbb{N}}$ of $H^n(K^\cdot)$ as follows. By definition of filtration, the inclusions $F^pK^n \to K^n$ for all n induce a map of $F^pK^\cdot \to K^\cdot$ of complexes, hence a map $H^n(F^pK^\cdot) \to H^n(K^\cdot)$ for all n. We define $F^pH^n(K^\cdot)$ to be the image of this map. Since $F^{p+1}K^\cdot \to F^pK^\cdot$ is a map of complexes, and since $F^0K^\cdot = K^\cdot$, we have

$$H^n(K^\cdot) = F^0H^n(K^\cdot) \supseteq F^1H^n(K^\cdot) \supseteq \dots$$

for all n. Moreover, for all n there is a p such that $F^pK^n = 0$, which gives $F^pH^n(K^\cdot) = 0$ (for the same p).

The main theorem of this handout is the following.

Theorem. *Let* $(K^{\cdot}, D)$ *be a filtered complex. Assume that* $F^pK^n = 0$ *for all* $n \in \mathbb{N}$ *and all* $p > n$. *For all* $r, n, p \in \mathbb{N}$, *define:*

$$\begin{aligned} X^{n;p}_{-1} &= F^pK^n \,, \\ X^{n;p}_r &= F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}) \,, \\ Y^{n;p}_r &= D(X^{n-1;p-(r-1)}_{r-1}) + X^{n;p+1}_{r-1} \,, \qquad \text{and} \\ E^{n;p}_r &= X^{n;p}_r / Y^{n;p}_r \,. \end{aligned}$$

Then:

(a). $Y^{n;p}_r \subseteq X^{n;p}_r$ *(and therefore* $E^{n;p}_r$ *is well defined) for all* r, n, p;

(b). D *induces well-defined maps*

$$d_r = d^{n;p}_r \colon E^{n;p}_r \to E^{n+1;p+r}_r$$

for all r, n, p;

(c). *with the above differentials, and letting* $E^{p,q}_r = E^{n;p}_r$ *and* $d^{p,q}_r = d^{n;p}_r$ *for all* r, n, p, q *with* $p + q = n$, $\{E_r, d_r\}_{r \ge 0}$ *is a spectral sequence; and*

(d). *we have* $F^{n+1}H^n(K^{\cdot}) = 0$ *for all* n, *and*

$$F^pH^n(K^{\cdot})/F^{p+1}H^n(K^{\cdot}) \cong E^{n;p}_\infty$$

for all $n \in \mathbb{N}$ *and all* $p = 0, \dots, n$.

Proof. **(a).** When $r = 0$,

$$\begin{aligned} X^{n;p}_0 &= F^pK^n \cap D^{-1}(F^pK^{n+1}) = F^pK^n \qquad \text{and} \\ Y^{n;p}_0 &= D(F^{p+1}K^{n-1}) + F^{p+1}K^n = F^{p+1}K^n \,, \end{aligned} \tag{1}$$

so clearly $Y^{n;p}_0 \subseteq X^{n;p}_0$.

Now assume $r > 0$. Since $X^{n-1;p-r+1}_{r-1} = F^{p-r+1}K^{n-1} \cap D^{-1}(F^pK^n)$, we have

$$D(X^{n-1;p-r+1}_{r-1}) \subseteq D(D^{-1}(F^pK^n)) \subseteq F^pK^n \,;$$

combining this with $D(X^{n-1;p-r+1}_{r-1}) \subseteq D^{-1}(F^{p+r}K^{n+1})$ (since $D \circ D = 0$) gives

$$D(X^{n-1;p-r+1}_{r-1}) \subseteq F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}) = X^{n;p}_r \,. \tag{2}$$

Also,

$$X^{n;p+1}_{r-1} = F^{p+1}K^n \cap D^{-1}(F^{p+r}K^{n+1}) \subseteq F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}) = X^{n;p}_r \,. \tag{3}$$

Combining (2) and (3) then gives

$$Y^{n;p}_r = D(X^{n-1;p-(r-1)}_{r-1}) + X^{n;p+1}_{r-1} \subseteq X^{n;p}_r \,.$$

(b). This amounts to checking that $D(X_r^{n;p}) \subseteq X_r^{n+1;p+r}$ and $D(Y_r^{n;p}) \subseteq Y_r^{n+1;p+r}$. For the first of these,

$$D(X_r^{n;p}) \subseteq D(D^{-1}(F^{p+r}K^{n+1})) \subseteq F^{p+r}K^{n+1}$$

because $X_r^{n;p} \subseteq D^{-1}(F^{p+r}K^{n+1})$ by definition, and

$$D(X_r^{n;p}) \subseteq D^{-1}(F^{p+2r}K^{n+2})$$

because $D \circ D = 0$, so

$$D(X_r^{n;p}) \subseteq F^{p+r}K^{n+1} \cap D^{-1}(F^{p+2r}K^{n+2}) = X_r^{n+1;p+r} .$$

As for $D(Y_r^{n;p})$,

$$\begin{aligned} D(Y_r^{n;p}) &= D(D(X_{r-1}^{n-1;p-(r-1)}) + X_{r-1}^{n;p+1}) \\ &= D(X_{r-1}^{n;p+1}) \\ &\subseteq D(X_{r-1}^{n;p+1}) + X_{r-1}^{n+1;p+r+1} \\ &= Y_r^{n+1;p+r} . \end{aligned}$$

Note that this holds also for $r = 0$, because the value of $X_{r-1}^{n+1;p+r+1}$ did not play a role here.

(c). This is a matter of checking that $d_r^2 = 0$ and that $H(E_r) = E_{r+1}$.

The fact that $d_r^2 = 0$ is immediate from the fact that $D^2 = 0$.

To check that $H(E_r) = E_{r+1}$, we follow Vakil 1.7.13.

Claim. $\ker d_r^{n;p} = \dfrac{X_{r-1}^{n;p+1} + X_{r+1}^{n;p}}{Y_r^{n;p}}$.

Proof. It is easy to check that $\ker d_r^{n;p} = (X_r^{n;p} \cap D^{-1}(Y_r^{n+1;p+r}))/Y_r^{n;p}$, so it suffices to show that

$$X_r^{n;p} \cap D^{-1}(Y_r^{n+1;p+r}) = X_{r-1}^{n;p+1} + X_{r+1}^{n;p} . \tag{4}$$

Indeed, we have

$$X_{r-1}^{n;p+1} = F^{p+1}K^n \cap D^{-1}(F^{p+r}K^{n+1}) \subseteq F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}) = X_r^{n;p} \tag{5}$$

and

$$\begin{aligned} &X_r^{n;p} \cap D^{-1}(X_{r-1}^{n+1;p+r+1}) \\ &\quad = F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}) \cap D^{-1}(F^{p+r+1}K^{n+1} \cap D^{-1}(F^{p+2r}K^{n+2})) \\ &\quad = F^pK^n \cap D^{-1}(F^{p+r}K^{n+1}) \cap D^{-1}(F^{p+r+1}K^{n+1}) \\ &\quad = F^pK^n \cap D^{-1}(F^{p+r+1}K^{n+1}) \\ &\quad = X_{r+1}^{n;p} . \end{aligned} \tag{6}$$

Therefore, by definition of $Y_r^{n+1;p+r}$, general properties of homomorphisms, (5), and (6), we have

$$\begin{aligned} X_r^{n;p} \cap D^{-1}(Y_r^{n+1;p+r}) &= X_r^{n;p} \cap D^{-1}(D(X_{r-1}^{n;p+1}) + X_{r-1}^{n+1;p+r+1}) \\ &= X_r^{n;p} \cap (X_{r-1}^{n;p+1} + D^{-1}(X_{r-1}^{n+1;p+r+1})) \\ &= X_{r-1}^{n;p+1} + (X_r^{n;p} \cap D^{-1}(X_{r-1}^{n+1;p+r+1})) \\ &= X_{r-1}^{n;p+1} + X_{r+1}^{n;p} \ . \end{aligned}$$

This is (4), so the claim is proved. $\square$

Now consider the image of $d_r^{n;p-r} \colon E_r^{n;p-r} \to E_r^{n;p}$. Since $X_{r-1}^{n-1;p-r+1} \subseteq X_r^{n-1;p-r}$,

$$\begin{aligned} \operatorname{im} d_r^{n;p-r} &= \frac{D(X_r^{n-1;p-r}) + Y_r^{n;p}}{Y_r^{n;p}} \\ &= \frac{D(X_r^{n-1;p-r}) + D(X_{r-1}^{n-1;p-r+1}) + X_{r-1}^{n;p+1}}{Y_r^{n;p}} \\ &= \frac{D(X_r^{n-1;p-r}) + X_{r-1}^{n;p+1}}{Y_r^{n;p}} \ . \end{aligned}$$

It will suffice to show that there is a well-defined isomorphism

$$E_{r+1}^{n;p} = \frac{X_{r+1}^{n;p}}{Y_{r+1}^{n;p}} = \frac{X_{r+1}^{n;p}}{D(X_r^{n-1;p-r}) + X_r^{n;p+1}} \xrightarrow{\phi} \frac{X_{r+1}^{n;p} + X_{r-1}^{n;p+1}}{D(X_r^{n-1;p-r}) + X_{r-1}^{n;p+1}} \cong \frac{\ker d_r^{n;p}}{\operatorname{im} d_r^{n;p-r}} \ .$$

We first claim that

$$X_{r+1}^{n;p} \cap (D(X_r^{n-1;p-r}) + X_{r-1}^{n;p+1}) = D(X_r^{n-1;p-r}) + X_r^{n;p+1} \ . \tag{7}$$

Since $D(X_r^{n-1;p-r}) \subseteq D(D^{-1}(F^pK^n))$ and $D \circ D = 0$, we have

$$D(X_r^{n-1;p-r}) \subseteq F^pK^n \cap D^{-1}(0) \subseteq F^pK^n \cap D^{-1}(F^{p+r+1}K^{n+1}) = X_{r+1}^{n;p} \ .$$

Also

$$\begin{aligned} X_{r+1}^{n;p} \cap X_{r-1}^{n;p+1} &= F^pK^n \cap D^{-1}(F^{p+r+1}K^{n+1}) \cap F^{p+1}K^n \cap D^{-1}(F^{p+r}K^{n+1}) \\ &= F^{p+1}K^n \cap D^{-1}(F^{p+r+1}K^{n+1}) \\ &= X_r^{n;p+1} \ . \end{aligned}$$

Combining these two facts gives (7), because

$$\begin{aligned} X_{r+1}^{n;p} \cap (D(X_r^{n-1;p-r}) + X_{r-1}^{n;p+1}) &= D(X_r^{n-1;p-r}) + (X_{r+1}^{n;p} \cap X_{r-1}^{n;p+1}) \\ &= D(X_r^{n-1;p-r}) + X_r^{n;p+1} \ . \end{aligned}$$

Therefore ϕ is well-defined and injective. Surjectivity of ϕ is clear, so ϕ is an isomorphism and (c) is proved.

(d). First of all, the fact that $F^{n+1}H^n(K^\cdot) = 0$ follows immediately from the assumption that $F^{n+1}K^n = 0$.

By definition of $F^pH^n(K^\cdot)$,

$$F^pH^n(K^\cdot) = \operatorname{im}\left(H^n(F^pK^\cdot) \longrightarrow \frac{\ker D^n}{\operatorname{im} D^{n-1}}\right) = \frac{(F^pK^n \cap \ker D) + \operatorname{im} D}{\operatorname{im} D} \; .$$

First consider $E_\infty^{n;p}$. Since $d_r^{n;p} = 0$ for all $r > n+1$ (since $E_r^{n;p+r} = 0$), we have $X_r^{n;p} = X_{r+1}^{n;p}$ and $Y_r^{n;p} = Y_{r+1}^{n;p}$ for all $r > n+1$; call these groups $X_\infty^{n;p}$ and $Y_\infty^{n;p}$, respectively. We have

$$X_\infty^{n;p} = F^pK^n \cap \ker D$$

and

$$Y_r^{n;p} = D(F^{p-r+1}K^{n-1} \cap D^{-1}(F^pK^n)) + X_{r-1}^{n;p+1}$$

for all n, r, and p, so

$$\begin{aligned} Y_\infty^{n;p} &= D(K^{n-1} \cap D^{-1}(F^pK^n)) + X_\infty^{n;p+1} \\ &= D(D^{-1}(F^pK^n)) + X_\infty^{n;p+1} \\ &= (F^pK^n \cap \operatorname{im} D) + (F^{p+1}K^n \cap \ker D) \; . \end{aligned}$$

And, naturally, $E_\infty^{n;p} = X_\infty^{n;p}/Y_\infty^{n;p}$.

We then claim that there is a well-defined isomorphism

$$\begin{aligned} E_\infty^{n;p} &= \frac{F^pK^n \cap \ker D}{(F^pK^n \cap \operatorname{im} D) + (F^{p+1}K^n \cap \ker D)} \\ &\xrightarrow{\phi} \frac{(F^pK^n \cap \ker D) + \operatorname{im} D}{(F^{p+1}K^n \cap \ker D) + \operatorname{im} D} = \frac{F^pH^n(K)}{F^{p+1}H^n(K)} \; . \end{aligned}$$

To see this, we first note that

$$\begin{aligned} F^pK^n \cap \ker D \cap ((F^{p+1}K^n \cap \ker D) + \operatorname{im} D) &= F^pK^n \cap ((F^{p+1}K^n \cap \ker D) + \operatorname{im} D) \\ &= (F^{p+1}K^n \cap \ker D) + (F^pK^n \cap \operatorname{im} D) \; . \end{aligned}$$

Indeed, the first step holds because $\operatorname{im} D \subseteq \ker D$, and so $\ker D$ contains the quantity in parentheses. The second step is true because $F^{p+1}K^n \cap \ker D$ is contained in F^pK^n.

Therefore ϕ is well-defined and injective. It is clearly surjective, so it is the desired isomorphism. $\square$