Math 254B. Homework #4
Due Wednesday 3 March, online

1. Let \( X \subseteq \mathbb{P}_Q^2 \times \mathbb{P}_Q^1 \) be the closed subscheme defined by \( x_0 y_1 = x_1 y_0 \), where \([x_0 : x_1 : x_2] \) and \([y_0 : y_1] \) are homogeneous coordinates on \( \mathbb{P}_Q^2 \) and \( \mathbb{P}_Q^1 \), respectively. Let \( \phi : X \to \mathbb{P}_Q^2 \) be the restriction of the canonical projection \( \mathbb{P}_Q^2 \times \mathbb{P}_Q^1 \to \mathbb{P}_Q^2 \), and let \( \psi : X \to \mathbb{P}_Q^5 \) be the composite function \( X \to \mathbb{P}_Q^2 \times \mathbb{P}_Q^1 \to \mathbb{P}_Q^5 \), where the second map is the Segre embedding

\[
([x_0 : x_1 : x_2], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1 : x_2 y_0 : x_2 y_1].
\]

This map is also a closed embedding (you don’t need to prove this), and therefore so is \( \psi \). Therefore we regard \( X \) as a closed subscheme of \( \mathbb{P}_Q^5 \) via \( \psi \), and let \( h_Q(P) = h_Q(\psi(P)) \) for all \( P \in X(\mathbb{Q}) \).

Note that \( \phi \) is injective (at least on closed points), except over the point \([0 : 0 : 1]\), and that the fiber \( \phi^{-1}([0 : 0 : 1]) \) is isomorphic to \( \mathbb{P}_Q^1 \). The map \( \phi \) is called the **blowing-up of \( \mathbb{P}_Q^2 \) at the point \([0 : 0 : 1]\);** see Hartshorne I §4.

(a). Show that, for all \( P \in X(\mathbb{Q}) \),

\[
h_Q(\phi(P)) \leq h_Q(P) + O(1),
\]

where the implicit constant in \( O(1) \) is independent of \( P \).

(b). Show that the opposite inequality \( h_Q(P) \leq h_Q(\phi(P)) + O(1) \) is false, even for \( P \notin \phi^{-1}([0 : 0 : 1]) \).

2(NC). Let \( k \) be a number field or function field, and let \( f : X \to Y \) be a morphism of projective schemes over \( k \). Let \( \mathcal{M} \) be a line sheaf on \( Y \), and let \( \mathcal{L} = f^* \mathcal{M} \) be its pull-back to \( X \). Let \( h_{\mathcal{L}, k} \) be a height function for \( \mathcal{L} \) and \( k \) on \( X \).

For any given \( Q \in Y(k) \), we have \( \mathcal{L}|_{f^{-1}(Q)} \cong \mathcal{O}_{f^{-1}(Q)} \), so \( h_{\mathcal{L}, k} \) is bounded on \( f^{-1}(Q) \). These bounds cannot in general be taken to be independent of \( Q \), though, since usually \( h_{\mathcal{L}, k} \) is unbounded.

However, we also have \( h_{\mathcal{L}, k}(P_1) = h_{\mathcal{L}, k}(P_2) + O(1) \) for all \( P_1, P_2 \in f^{-1}(Q) \).

Show that this latter bound can be taken independent of \( Q \).

This problem will be graded, in part, on giving a proof that neatly addresses the reason why this is true (i.e., really “nails it”). (You will not be penalized for including more details, though.)

3. Let \( k \) be a field, let \( A_0 \) be the subring of \( k[x, y] \) generated by homogeneous polynomials of degree \( \neq 1 \), let \( A = (A_0)_{x-1} \), and let \( X = \text{Spec} A \).

(a). Show that \( X \) is a variety over \( k \) and is regular in codimension one.

(b). Show that \( X \) is not normal.

(c). Show that the divisor \((x-1)\) equals zero as a Weil divisor, but that \( x-1 \) is not a regular function on \( X \). (Note that \( x-1 \) is an element of \( K(X) = k(x, y) \).)