Math 254B. Homework #4

Due Wednesday 3 March, online

1. Let $X \subseteq \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ be the closed subscheme defined by $x_0 y_1 = x_1 y_0$, where $[x_0 : x_1 : x_2]$ and $[y_0 : y_1]$ are homogeneous coordinates on $\mathbb{P}^2_{\mathbb{Q}}$ and $\mathbb{P}^1_{\mathbb{Q}}$, respectively. Let $\phi\colon X \to \mathbb{P}^2_{\mathbb{Q}}$ be the restriction of the canonical projection $\mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Q}}$, and let $\psi\colon X \hookrightarrow \mathbb{P}^5_{\mathbb{Q}}$ be the composite function $X \longrightarrow \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^5_{\mathbb{Q}}$, where the second map is the Segre embedding

$$([x_0 : x_1 : x_2], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1 : x_2 y_0 : x_2 y_1] \ .$$

This map is also a closed embedding (you don't need to prove this), and therefore so is ψ. Therefore we regard X as a closed subscheme of $\mathbb{P}^5_{\mathbb{Q}}$ via ψ, and let $h_{\mathbb{Q}}(P) = h_{\mathbb{Q}}(\psi(P))$ for all $P \in X(\overline{\mathbb{Q}})$.

Note that ϕ is injective (at least on closed points), except over the point $[0 : 0 : 1]$, and that the fiber $\phi^{-1}([0 : 0 : 1])$ is isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$. The map ϕ is called the **blowing-up of $\mathbb{P}^2_{\mathbb{Q}}$ at the point** $[0 : 0 : 1]$; see Hartshorne I § 4.

(a). Show that, for all $P \in X(\overline{\mathbb{Q}})$,

$$h_{\mathbb{Q}}(\phi(P)) \le h_{\mathbb{Q}}(P) + O(1) \ ,$$

where the implicit constant in $O(1)$ is independent of P.

(b). Show that the opposite inequality $h_{\mathbb{Q}}(P) \le h_{\mathbb{Q}}(\phi(P)) + O(1)$ is false, even for $P \notin \phi^{-1}([0 : 0 : 1])$.

2(NC). Let k be a number field or function field, and let $f\colon X \to Y$ be a morphism of projective schemes over k. Let $\mathscr{M}$ be a line sheaf on Y, and let $\mathscr{L} = f^*\mathscr{M}$ be its pull-back to X. Let $h_{\mathscr{L},k}$ be a height function for $\mathscr{L}$ and k on X.

For any given $Q \in Y(\bar{k})$, we have $\mathscr{L}\big|_{f^{-1}(Q)} \cong \mathscr{O}_{f^{-1}(Q)}$, so $h_{\mathscr{L},k}$ is bounded on $f^{-1}(Q)$. These bounds cannot in general be taken to be independent of Q, though, since usually $h_{\mathscr{L},k}$ is unbounded.

However, we also have $h_{\mathscr{L},k}(P_1) = h_{\mathscr{L},k}(P_2) + O(1)$ for all $P_1, P_2 \in f^{-1}(Q)$.

Show that this latter bound can be taken independent of Q.

This problem will be graded, in part, on giving a proof that neatly addresses the reason why this is true (i.e., really "nails it"). (You will not be penalized for including more details, though.)

3. Let k be a field, let A_0 be the subring of $k[x, y]$ generated by homogeneous polynomials of degree $\neq 1$, let $A = (A_0)_{x^2-1}$, and let $X = \operatorname{Spec} A$.

(a). Show that X is a variety over k and is regular in codimension one.

(b). Show that X is not normal.

(c). Show that the divisor $(x-1)$ equals zero as a Weil divisor, but that $x-1$ is not a regular function on X. (Note that $x-1$ is an element of $K(X) = k(x, y)$.)