Math 254B. Homework #2
Due Wednesday 10 February, online

1. State and prove a variation of Roth’s theorem, in a style similar to one of the variants of Roth’s theorem in the second question on the first homework assignment, but with \( \alpha \in \mathbb{Q} \) and with a sharper inequality.

2. For nonzero \( a \in \mathbb{Q} \), the morphism \( \phi_a : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Q}} \) given by

\[
[t : u] \mapsto [at^2 : tu : u^2]
\]

gives an isomorphism from \( \mathbb{P}^1_{\mathbb{Q}} \) to the curve \( X_a \subseteq \mathbb{P}^2_{\mathbb{Q}} \) defined by \( xz = ay^2 \). The heights of rational points are related by

\[
h_{\mathbb{Q}}(\phi_a(P)) = 2h_{\mathbb{Q}}(P) + O(1)
\]

for all \( P \in \mathbb{P}^1(\mathbb{Q}) \) (don’t worry about points rational over larger fields). Here the constant \( O(1) \) depends on \( a \). Determine upper and lower bounds for the \( O(1) \) term, as functions of \( a \in \mathbb{Q} \).