# CHAPTER 3
# MODELS

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Following the observation that the field of functions of a curve $Y$ shares many diophantine properties with number fields, it was discovered that if $W$ is a variety over $k$ and $\pi\colon X \to Y$ is a morphism with generic fiber isomorphic to $W$, then many function-field counterparts to number-theoretic questions could be framed (and sometimes solved) in terms of geometry of $X$ and $\pi$. This then led to a desire to apply these methods to varieties over number fields, giving rise to the study of models and ultimately to Arakelov theory.

The present chapter introduces models and discusses some of their properties, paving the way for the treatment of arithmetic schemes and Arakelov theory in the next few chapters.

An advantage of switching over to arithmetic schemes and Arakelov theory is that the geometric objects allow us to work with exact quantities, instead of functions up to $O(1)$; see Remark 5.5.

## §1. Definitions

It will be useful to discuss models not only over global objects, but also over objects corresponding to just one place of a given field.

We first define the base object, also called a model since its definition is (circularly) compatible with the definition of a model of a variety.

**Definition 1.1.** Let $k$ be a global field.

(a). If $k$ is a function field with field of constants $F$, then the **maximal model** $Y$ for $k$ is the nonsingular projective curve over $F$ with function field $k$. After fixing the isomorphism $K(Y) \cong k$, the maximal model is uniquely determined up to unique isomorphism.

(b). If $k$ is a number field, then the **maximal model** for $k$ is $\operatorname{Spec} R$, where $R$ is the ring of integers of $k$.

(c). A **model** for $k$ is a nonempty Zariski-open subset of its maximal model.

(d). Note that the set of non-archimedean places of $k$ corresponds bijectively to the set of closed points of its maximal model. If $v$ is a non-archimedean place of $k$, and $Y$ is the maximal model of $k$, then the **local model** for $k$ at $v$ is defined to be the spectrum of the local ring of $Y$ at the closed point corresponding to $v$.

In the context of this definition, the word *model*, by itself, will always mean a model as in part (c); if this is to be emphasized, then we may say *global model* instead.

**Definition 1.2.** Let $k$ be a non-archimedean local field. Then the **maximal model** for $k$ is the spectrum of its valuation ring, and a **model** for $k$ is a nonempty Zariski-open subset of its maximal model. A **local model** for $k$ (at its unique place) is its maximal model.

**Definition 1.3.** Let $k$ be a global or non-archimedean local field, and let $W$ be a scheme of finite type over $k$. Let $Y$ be a model for $k$ (resp. a local model for $k$ at a non-archimedean place $v$). Then a **(global) model** for $W$ over $Y$ (resp. a **local model** for $W$ at $v$) is a flat morphism $\pi\colon X \to Y$ of finite type, together with an isomorphism $i\colon W \xrightarrow{\sim} X_k := X \times_Y \operatorname{Spec} k$ from $W$ to the generic fiber of $\pi$. A model or local model is **proper** or **projective** if $\pi$ is proper or projective, respectively.

Throughout this chapter, if $Y$ is a model or local model for $k$ and $X$ is a scheme over $Y$, then $X_k$ denotes the generic fiber $X \times_Y \operatorname{Spec} k$. In applications, one often starts with a flat morphism $\pi\colon X \to Y$ of finite type, which is then a model for its own generic fiber $X_k$. Or, by abuse of notation, one can start with a scheme $X_k$ of finite type over $k$, and then choose a model $X$ for $X_k$ (by the results of Section 2).

In contrast to the situation with Definition 1.1, the word *model* in this context may refer to either a global model or a local model: if $Y$ is a local model for $k$ at $v$, and if $X$ is a local model for $W$ at $v$, mapping to $Y$, then we will also say that $X$ is a model for $W$ over $Y$. The fact that $X$ is a *local* model in this case is already implied by the fact that $Y$ is a local model.

Note also that if $k$ is a (non-archimedean) local field, then *model* and *local model* are both defined, but the definitions amount to the same thing.

**Remark 1.4.** If $W$ is a variety, then by **0:**3.11, any model $X$ for $W$ is integral. Conversely, if $k$ and $Y$ are as in Definition 1.3, if $W$ is a variety over $k$, if $\pi\colon X \to Y$ is a morphism of finite type whose generic fiber is isomorphic to $W$, and if $X$ is integral, then $X$ has only one associated point, so **0:**3.11 implies that $\pi$ is flat. Thus in that case $X$ is a model for $W$.

**Remark 1.5.** Unless it is proper (or projective), nothing in the definition of a model requires that $\pi$ be surjective. See Examples 1.7 and 1.9.

**Proposition 1.6.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $W$ be a variety over $k$, and let $X$ be a model for $W$*

*over $Y$. If $Y$ is a local model, then assume also that $X$ is proper over $Y$. Then* $\dim X = \dim W + 1$.

*Proof.* Immediate from Proposition **0:**2.6b. □

**Example 1.7.** Let $W$ be a scheme of finite type over a global field $k$.

(a). Let $Y$ be a model for $k$. Then there exists a morphism $\pi\colon W \to Y$. This morphism is flat, but it is not of finite type, so $W$ is not a model for itself over $Y$.

(b). If $Y$ is a local model for $k$ at some non-archimedean place $v$, however, then $W$ is a model for itself over $Y$, since in that case $\pi$ is of finite type. (This is what you would get by taking any local model and removing the closed fiber.) This shows that Proposition 1.6 is false without the assumption on proper models.

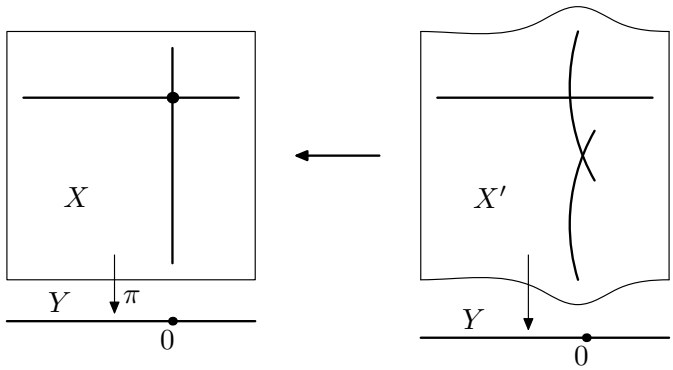
(c). Let $Y$ be a model for $k$, and let $v$ be a non-archimedean place of $k$ corresponding to a closed point of $Y$, also denoted $v$. Let $Y_v = \operatorname{Spec} \mathscr{O}_{Y,v}$ be the spectrum of the valuation ring of $v$, and note that there is a morphism $Y_v \to Y$. Then any model $X$ for $W$ over $Y$ can be localized at $v$ to give a local model $X \times_Y Y_v$ for $W$ over $Y_v$.

**Definition 1.8.** Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $W$ be a scheme of finite type over $k$, and let $\pi_1\colon X_1 \to Y$ and $\pi_2\colon X_2 \to Y$ be models for $W$ over $Y$. We say that $X_1$ **dominates** $X_2$ if there is a morphism $f\colon X_1 \to X_2$ over $Y$ compatible with the isomorphisms of $W$ with the generic fibers of $\pi_1$ and $\pi_2$:

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dr] & \\
X_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\
 & Y &
\end{tikzcd}$$

**Example 1.9.** Let $k = \mathbb{C}(t)$, with model $Y := \mathbb{A}^1_{\mathbb{C}}$, and let $W = \mathbb{A}^1_k$. Then $X := \mathbb{A}^2_{\mathbb{C}}$ is a model for $W$ over $Y$, with $\pi\colon \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ given by projection to the first coordinate. We can think of $X$ as an open subscheme of $\mathbb{P}^1_Y = \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$. Let $X'$ be the blowing-up of the point $(\infty, 0)$ on $\mathbb{P}^1_Y$; it is also a model for $\mathbb{P}^1_k$ over $Y$, and $X$ is still isomorphic

to an open subset of it since the point blown up was not in $X$.




In this new model for $\mathbb{P}^1_k$, the fiber over $0 \in Y$ has two irreducible components, one of which is "missing" from $X$. If one put this irreducible component back (except for the point where it intersects the strict transform of the curve $\mathbb{A}^1 \times \{\infty\}$), and removed the other irreducible component, this would result in another model for $\mathbb{A}^1_k$, which also maps surjectively to $Y$. However, the intersection of these two models lacks a fiber over $0 \in Y$. Intersection is important because any model that dominates the two models would also have to dominate their intersection. This illustrates why we allow models to be missing a whole fiber.

We end this section with some general results about models.

**Lemma 1.10.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $\pi' \colon X' \to Y$ be a flat morphism of finite type. Let $i \colon W \hookrightarrow X'_k$ be a closed subscheme of the generic fiber. Finally, let $X$ be the scheme-theoretic image of $i$ in $X'$. Then $X \to Y$ is a model for $W$.*

*Proof.* Since $X$ is a closed subscheme of $X'$, it is of finite type over $Y$.

Clearly $i$ factors through $X_k$. The resulting map $W \to X_k$ is again a closed immersion. To show that it is an isomorphism, we may assume that $X'$ and $Y$ are affine (since scheme-theoretic image is defined locally); say $X' = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then $X$ is defined by the kernel $\mathfrak{a}$ of the map $A \to \Gamma(W, \mathscr{O}_W)$, and $X_k$ is defined by the ideal $\mathfrak{a} \otimes_B k$ in $A \otimes_B k$, since $k$ is a localization of $B$. Since $\Gamma(W, \mathscr{O}_W)$ is a $k$-algebra, the map $A \to \Gamma(W, \mathscr{O}_W)$ factors uniquely through $A \otimes_B k$, so $X_k$ is the scheme-theoretic image of $i \colon W \to X'_k$. Since $i$ is a closed immersion, it induces an isomorphism of $W$ with $X_k$.

By **0:**3.11, it then follows that $X \to Y$ is flat, hence $X$ is a model for $W$. □

**Proposition 1.11.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $W$ be a scheme of finite type over $k$, and let $X$ be a model for $W$ over $Y$. Then $X_{\mathrm{red}}$ is a model for $W_{\mathrm{red}}$ over $Y$.*

*Proof.* Let $X'$ be the scheme-theoretic image of $W_{\mathrm{red}}$ in $X$. Since $X_{\mathrm{red}}$ is a closed subscheme of $X$ containing the image of $W_{\mathrm{red}}$, $X'$ is a closed subscheme of $X_{\mathrm{red}}$.

But the topological space underlying $X'$ cannot be strictly smaller than the space of $X_{\mathrm{red}}$, since the generic point of each irreducible component of $X_{\mathrm{red}}$ is also the generic point of an irreducible component of $X$, so it must lie on $X_k$ by Proposition **0:**3.11 (and the fact that primary decomposition is preserved by localizing). Thus it lies on $X'$. Therefore $X'$ and $X_{\mathrm{red}}$ must have the same underlying topological space, so they are equal.

By Lemma 1.10, $X_{\mathrm{red}} = X'$ is therefore a model for $W_{\mathrm{red}}$. □

**Proposition 1.12.** *Let $W$, $k$, $Y$, and $X$ be as in Proposition 1.11. Then the irreducible components of $X$ are in bijection with the irreducible components of $W$ (given by restriction to the generic fiber). Also, if $X'$ and $W'$ are corresponding irreducible components of $X$ and $W$, respectively (with reduced induced subscheme structures), then $X'$ is a model for $W'$.*

*Proof.* By Proposition **0:**3.11, all associated points of $X$ lie on its generic fiber, which we identify with $W$. This gives the bijection of irreducible components. Let $W'$ be an irreducible component of $W$, as in the statement of the proposition, and let $X''$ be the scheme-theoretic image of the map $W' \to X$. Then $X''$ is a closed integral subscheme of $X$ (by Proposition **0:**3.11, again) containing the generic point of $W'$. Therefore we must have $X'' = X'$, and Lemma 1.10 then implies that $X''$ is a model for $W'$. □

## §2. Existence of models

This section shows that models always exist. It also shows that the extension of a model may be chosen such that certain given coherent sheaves, vector sheaves, line sheaves, or Cartier divisors on the variety extend to the same sort of objects on the model. In light of Example 1.7b, this section will emphasize global models over local models.

### Projective models

We start with the results for projective models, since they are much easier than the proper or general case.

**Proposition 2.1.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $W$ be a projective scheme over $k$. Then there exists a projective model $X$ for $W$ over $Y$. Moreover, given any very ample line sheaf or very ample Cartier divisor on $W$, the model can be chosen such that this line sheaf or Cartier divisor extends to the model (as a very ample line sheaf or very ample Cartier divisor, respectively).*

*Proof.* Let $\mathscr{O}(1)$ be a very ample line sheaf on $W$ and let $i\colon W \hookrightarrow \mathbb{P}^n_k$ be a corresponding projective embedding. Viewing $\mathbb{P}^n_k$ as the generic fiber of $\mathbb{P}^n_Y \to Y$, we can let $X$ be the scheme-theoretic image of $i$ in $\mathbb{P}^n_Y$. By Lemma 1.10, $X$ is a model for $W$ over $Y$. It is also easy to see that $X$ is a projective model, and that $\mathscr{O}(1)$ extends to $X$ as a very ample line sheaf over $Y$. Thus the assertion about extending a very ample line sheaf holds; the same holds for extending a very ample Cartier divisor $D$ (by working with $\mathscr{O}(D)$). □

**Proposition 2.2.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $W_1$ and $W_2$ be schemes of finite type over $k$. Let $X_1$ and $X_2$ be models for $W_1$ and $W_2$ over $Y$, respectively, and let $i_1\colon W_1 \to X_1$ and $i_2\colon W_2 \to X_2$ be maps inducing the given isomorphisms with the respective generic fibers. Let $f\colon W_1 \to W_2$ be a morphism over $k$, and let $X'$ be the scheme-theoretic closure of the image of $(i_1, i_2 \circ f)\colon W_1 \to X_1 \times_Y X_2$. Then $X'$ is a model for $W_1$ over $Y$ which dominates $X_1$, and $f$ extends to a morphism $g\colon X' \to X_2$ over $Y$. Moreover, if both $X_1$ and $X_2$ are proper (resp. projective), then so is $X'$.*

*Proof.* Since $W_2$ is separated over $k$, the map $(i_1, i_2 \circ f)\colon W_1 \to X_1 \times_Y X_2$ is a closed immersion into the generic fiber. Hence $X'$ is a model for $W_1$, by Lemma 1.10. Clearly it dominates $X_1$, and the projection to $X_2$ gives a morphism extending $f$. The last assertion is also trivial. □

**Corollary 2.3.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $W$ be a scheme of finite type over $k$. Then, for any two models $X_1$ and $X_2$ for $W$ over $Y$, there is a third model $X'$ which dominates both of them. Moreover, if both $X_1$ and $X_2$ are proper (resp. projective), then so is $X'$.*

*Proof.* Apply Proposition 2.2 to the identity morphism on $W$. □

**Corollary 2.4.** *Let $k$, $Y$, and $W$ be as in Proposition 2.1. Then, given any finite collections $\mathscr{L}_1, \dots, \mathscr{L}_r$ of line sheaves and $D_1, \dots, D_s$ of Cartier divisors on $W$, there exists a projective model $X$ for $W$ over $Y$ such that $\mathscr{L}_1, \dots, \mathscr{L}_r$ and $D_1, \dots, D_s$ extend to $X$ as line sheaves and Cartier divisors, respectively.*

*Proof.* Since any line sheaf or Cartier divisor may be written as a difference of two very ample line sheaves or Cartier divisors, respectively, we may assume that all of the $\mathscr{L}_i$ and $D_j$ are very ample. For each such $\mathscr{L}_i$ or $D_j$ choose a suitable projective model for $W$ such that $\mathscr{L}_i$ or $D_j$ extends, and let $X$ be a projective model that dominates all of these models. □

Note that we no longer require that very ample divisors or line sheaves remain very ample after extending. I suspect that this is not possible, but finding a counterexample is difficult since one has to show that all possible models lack the property.

**Proposition 2.5.** *Let $k$, $Y$, and $W$ be as in Proposition 2.1. Let $v \in Y$ be a closed point, let $Y_v = \operatorname{Spec} \mathscr{O}_{Y,v}$, and let $X_v$ be a projective model for $W$ over $Y_v$. Then there exists a projective model $X$ for $W$ over $Y$ such that $X_v \cong X \times_Y Y_v$ (compatible with the isomorphisms of the generic fibers). Moreover, given any finite lists $\mathscr{L}_1, \dots, \mathscr{L}_r$ and $D_1, \dots, D_s$ of line sheaves and Cartier divisors, respectively, on $X_v$, the model $X$ may be chosen such that $\mathscr{L}_1, \dots, \mathscr{L}_r$ and $D_1, \dots, D_s$ extend to line sheaves and Cartier divisors, respectively, on $X$.*

*Proof.* By Corollary 2.3, we reduce immediately to the case of one line sheaf $\mathscr{L}$ or one Cartier divisor $D$, assumed to be very ample over $Y_v$. Let $i\colon X_v \hookrightarrow \mathbb{P}^n_{Y_v}$ be a closed

immersion corresponding to $\mathscr{L}$ or $D$, and let $X$ be the scheme-theoretic image of the composition $W \to X_v \to \mathbb{P}^n_{Y_v} \to \mathbb{P}^n_Y$. Then $X$ is a projective model for $W$, and the projective embedding of $X$ gives an extension of $\mathscr{L}$ or $D$, respectively.

To obtain the remaining assertions, it will suffice to show that $X_v \cong X \times_Y Y_v$. Both are closed subschemes of $\mathbb{P}^n_{Y_v}$, both have the same generic fiber, and neither of them has any associated points outside of the generic fiber. Therefore by Proposition **0:**3.11 they are both isomorphic to the scheme-theoretic image of $W$ in $\mathbb{P}^n_{Y_v}$, hence they are isomorphic. □

**Models in general**

We now consider similar questions for models that are not required to be proper or projective.

We start with some general lemmas.

**Lemma 2.6.** *Let $f\colon X \to Y$ be a morphism of noetherian schemes. Assume that $Y$ can be covered by open affines $V_i = \operatorname{Spec} B_i$ such that, for each $i$, the scheme $f^{-1}(V_i)$ is isomorphic to $\operatorname{Spec} B_i[T^{-1}]$ over $V_i$ for some multiplicative subset $T$ of $B_i$. Then $f$ is scheme-theoretically dominant if and only if all associated points of $Y$ lie in the (set-theoretic) image of $f$.*

*Proof.* Since $f$ is a quasi-compact morphism, ([**Stacks**], 01R8, (3)) implies that $f$ scheme-theoretically dominant if and only if the sheaf map $\mathscr{O}_Y \to f_*\mathscr{O}_X$ is injective. This condition is local on $Y$, so it suffices to consider the case in which $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} B[T^{-1}]$ for some multiplicative subset $T$ of $B$. Then $f$ is scheme-theoretically dominant if and only if $B \to B[T^{-1}]$ is injective, which holds if and only if all elements of $T$ are nonzerodivisors. By ([**E**], Thm. 3.1b), this holds if and only if $T$ is disjoint from all associated primes of $B$, which holds if and only if all associated points of $\operatorname{Spec} B$ lie in the image of $f$. □

**Corollary 2.7.** *Let $f\colon X \to Y$ be a morphism of noetherian schemes, let $y \in Y$, and let $X_y = X \times_Y \mathscr{O}_{Y,y}$. Then $f$ is scheme-theoretically dominant if and only if all associated points of $X$ lie in the local fiber over $y$.*

*Proof.* This is immediate from Lemma 2.6. □

**Lemma 2.8.** *Let $Y$ be a noetherian scheme, let $y \in Y$, and let $X_y \to \operatorname{Spec} \mathscr{O}_{Y,y}$ be a separated scheme of finite type. Then there exist a scheme $X$, a morphism $\pi\colon X \to Y$ which is separated and of finite type, and an $\mathscr{O}_{Y,y}$-isomorphism $\iota\colon X_y \xrightarrow{\sim} X \times_Y \mathscr{O}_{Y,y}$. Also, $X$ may be chosen so that all of its associated points lie on $X \times_Y \mathscr{O}_{Y,y}$. Moreover, given vector sheaves $\mathscr{E}_{1,y}, \dots, \mathscr{E}_{r,y}$ and Cartier divisors $D_{1,y}, \dots, D_{s,y}$ on $X_y$, the above objects may be chosen such that there exist vector sheaves $\mathscr{E}_1, \dots, \mathscr{E}_r$ and Cartier divisors $D_1, \dots, D_s$ on $X$ such that $\iota^*\mathscr{E}_i \cong \mathscr{E}_{i,y}$ for all $i$ and $\iota^* D_j = D_{j,y}$ for all $j$.*

*Proof.* First note that if $Y'$ is an open subset of $Y$ containing $y$ and if $\pi'\colon X \to Y'$ satisfies the conditions of the lemma with $Y$ replaced by $Y'$, then the morphism

$\pi\colon X \to Y$ induced by $\pi'$ also satisfies the conditions of the lemma. Therefore we may freely shrink $Y$ by passing to an open subset containing $y$.

Cover $X_y$ with finitely many open affines $U_1, \dots, U_n$, assumed nonempty. Then $X_y$ can be recovered from the $U_i$ by glueing, according to the following glueing data: (i) open subsets $U_{ij} \subseteq U_i$ for all $i \neq j$, and (ii) $\mathscr{O}_{Y,y}$-isomorphisms $\phi_{ij}\colon U_{ij} \xrightarrow{\sim} U_{ji}$ for all $i \neq j$. These data satisfy:

(a). $\phi_{ij} = \phi_{ji}^{-1}$ for all $i \neq j$;
(b). $\phi_{ij}(U_{ij} \cap U_{i\ell}) = U_{ji} \cap U_{j\ell}$ for all distinct $i, j, \ell$; and
(c). $\phi_{i\ell} = \phi_{j\ell} \circ \phi_{ij}$ on $U_{ij} \cap U_{i\ell}$ for all distinct $i, j, \ell$.

It will suffice to extend the $U_i$ over $\operatorname{Spec} \mathscr{O}_{Y,y}$ to schemes $V_i$ over $Y$ that satisfy the same glueing conditions.

First of all, since $X_y$ is of finite type over $\mathscr{O}_{Y,y}$, the affine rings of the $U_i$ are finitely generated over $\mathscr{O}_{Y,y}$, say $U_i = \operatorname{Spec} \mathscr{O}_{Y,y}[t_1, \dots, t_{n_i}]/I_i$. After replacing $Y$ with an open affine neighborhood $\operatorname{Spec} B$ of $y$, we may assume that all of the ideals $I_i$ can be generated by polynomials in $B[t_1, \dots, t_{n_i}]$. For each $i$, let $V_i$ be the open subscheme of the noetherian scheme $\operatorname{Spec} B[t_1, \dots, t_{n_i}]/I_i$ associated to the complement of the union of the closures of all associated points of $B[t_1, \dots, t_{n_i}]/I_i$ whose closures do not meet $U_i$. Then $V_i$ is separated and of finite type over $Y$, $V_i \times_Y \mathscr{O}_{Y,y} \cong U_i$ (via obvious isomorphisms), and all associated points of $V_i$ lie in $V_i \times_Y \mathscr{O}_{Y,y}$. From now on, we identify the topological space of $U_i$ with a subspace of the topological space of $V_i$ for all $i$.

Since the topology on $U_i$ is induced by that on $V_i$, there are open subsets $V_{ij}$ of $V_i$ such that $V_{ij} \cap U_i = U_{ij}$ (for all $i \neq j$). It is again true that all associated points of $V_{ij}$ lie on $U_{ij}$.

The isomorphisms $\phi_{ij}$ extend to morphisms on open subsets of the $V_{ij}$ containing the local fibers $U_{ij}$; hence, by shrinking $Y$, we may assume that the $\phi_{ij}$ extend to morphisms $V_{ij} \to V_j$, also denoted $\phi_{ij}$. The complement of $\phi_{ij}^{-1}(V_{ji})$ in $V_{ij}$ is a closed subset disjoint from $U_{ij}$, so after shrinking $Y$ further we may assume that $\phi_{ij}$ maps $V_{ij}$ to $V_{ji}$. Finally, since all associated points of $V_{ij}$ lie on $U_{ij}$, Corollary 2.7 implies that the map $U_{ij} \to V_{ij}$ is scheme-theoretically dominant. Therefore, since $\phi_{ji} \circ \phi_{ij}$ coincides with the identity morphism on the local fiber $U_{ij}$, they coincide on all of $V_{ij}$ by Proposition **0:**3.12. Thus the $\phi_{ij}$ are isomorphisms $V_{ij} \xrightarrow{\sim} V_{ji}$.

The open sets $\phi_{ij}(V_{ij} \cap V_{i\ell})$ and $V_{ji} \cap V_{j\ell}$ are open subsets of $V_i$ that agree on the local fiber at $y$; hence by shrinking $Y$ outside of $y$ we may assume that they are equal. This ensures that condition (b) is satisfied.

The equality $\phi_{i\ell} = \phi_{j\ell} \circ \phi_{ij}$ holds on $V_{ij} \cap V_{i\ell}$ because it agrees on the local fiber, again by Proposition **0:**3.12.

Thus, we may glue the $V_i$ to get a scheme $X$ (not necessarily separated) whose local fiber over $\operatorname{Spec} \mathscr{O}_{Y,y}$ is isomorphic to $X_y$. All associated points of $X$ lie on $X \times_Y \mathscr{O}_{Y,y}$ because this is true locally on each $V_i$. Also, $X$ is of finite type over $Y$ because its open subsets $V_i$ are. The complement of the diagonal in $X \times_Y X$ in its closure is a closed subset not meeting the local fiber at $v$, so we may shrink $Y$ to ensure that $X$ is separated over $Y$. Thus $X$ satisfies the desired conditions.

The final assertion, concerning extending vector sheaves and Cartier divisors, is left as an exercise for the reader. □

**Corollary 2.9.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $W$ be a scheme of finite type over $k$. Then there exists a model $X$ for $W$ over $Y$. Moreover, given vector sheaves $\mathscr{E}_1, \dots, \mathscr{E}_r$ and Cartier divisors $D_1, \dots, D_s$ on $W$, the model $X$ may be chosen such that $\mathscr{E}_1, \dots, \mathscr{E}_r$ and $D_1, \dots, D_s$ extend to vector sheaves and Cartier divisors on $X$, respectively.*

*Proof.* This is immediate from Lemma 2.8. Indeed, we may take $y$ to be the generic point of $Y$, in which case the local ring $\mathscr{O}_{Y,y}$ is $k$. □

**Proper models**

Existence of proper models follows easily from the preceding results, via Nagata's embedding theorem.

**Lemma 2.10.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $W$ be a proper scheme over $k$. Then any model $X_0$ for $W$ over $Y$ extends to a proper model $X$ for $W$ over $Y$ (in such a way that that $X_0$ is isomorphic to a dense open subset of $X$). Moreover, given vector sheaves $\mathscr{E}_1, \dots, \mathscr{E}_r$ and Cartier divisors $D_1, \dots, D_s$ on $W$, the model $X$ may be chosen such that $\mathscr{E}_1, \dots, \mathscr{E}_r$ and $D_1, \dots, D_s$ extend to vector sheaves and Cartier divisors on $X$, respectively.*

*Proof.* By Nagata's embedding theorem (Theorem **B:**5.9), $X_0$ may be embedded as a schematically dense open subscheme of a proper scheme $X$ over $Y$. Moreover, $X$ can be chosen such that $\mathscr{E}_1, \dots, \mathscr{E}_r$ and $D_1, \dots, D_s$ extend to vector sheaves and Cartier divisors on $X$, respectively.

It remains only to check that $X$ is flat over $Y$, and that it has the right generic fiber. For flatness, $X$ contains no associated points outside of its generic fiber, by Propositions **0:**3.2 and **0:**3.11. By the latter proposition, $X$ is flat over $Y$.

Since $X_0$ is dense in $X$, the map $i\colon W \to X_k$ induces an isomorphism of $W$ with an open dense subscheme of $X_k$. Since $W$ is proper over $k$, the map $i$ is also proper, hence it has closed image. Thus it is an isomorphism, completing the proof that $X$ is a model for $W$ over $Y$. □

**Theorem 2.11.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, and let $W$ be a proper scheme over $k$. Then there exists a proper model $X$ for $W$ over $Y$. Moreover, given vector sheaves $\mathscr{E}_1, \dots, \mathscr{E}_r$ and Cartier divisors $D_1, \dots, D_s$ on $W$, the model $X$ may be chosen such that $\mathscr{E}_1, \dots, \mathscr{E}_r$ and $D_1, \dots, D_s$ extend to vector sheaves and Cartier divisors on $X$, respectively.*

*Proof.* Immediate from Corollary 2.9 and Lemma 2.10. □

**Theorem 2.12.** *Let $k$, $Y$, and $W$ be as in Theorem 2.11; let $v$ be a non-archimedean place of $k$ corresponding to a closed point $v \in Y$; let $Y_v = \operatorname{Spec} \mathscr{O}_{Y,v}$; and let $X_v$*

*be a proper local model for $W$ over $Y_v$. Then there exists a proper model $X$ for $W$ over $Y$ such that $X_v \cong X \times_Y Y_v$. Moreover, given vector sheaves $\mathscr{E}_1, \dots, \mathscr{E}_r$ and Cartier divisors $D_1, \dots, D_s$ on $X_v$, the model $X$ may be chosen such that $\mathscr{E}_1, \dots, \mathscr{E}_r$ and $D_1, \dots, D_s$ extend to vector sheaves and Cartier divisors on $X$, respectively.*

*Proof.* Immediate from Lemmas 2.8 and 2.10. □

**Splicing models**

If one has several (finitely many) models for the same scheme, each of which has desirable properties at disjoint sets of places, then these models can be combined so that the local fibers come from any of the given models. We start by comparing models.

**Lemma 2.13.** *Let $X_1$ and $X_2$ be schemes of finite type over a noetherian integral scheme $Y$. Let $k = K(Y)$, and let $\phi\colon X_{1,k} \to X_{2,k}$ be a morphism of the generic fibers. Then there exists a nonempty open subset $U \subseteq Y$ such that $\phi$ extends to a morphism $\psi\colon X_1 \times_Y U \to X_2 \times_Y U$.*

*Proof.* Since the set where $X_1$ is flat over $Y$ is Zariski-open ([**EGA**], IV 11.1.1) and contains the generic fiber, we may assume that $X_1$ is flat over $Y$, by restricting $Y$ to a nonempty Zariski-open subset. We may also assume that $Y$ is affine, say $Y = \operatorname{Spec} B$.

We first claim that, for every point $\xi$ in the generic fiber $X_{1,k}$, there is an open neighborhood of $\xi$ in $X_1$ such that $\phi$ extends uniquely to a morphism on that neighborhood. Indeed, let $V_1$ and $V_2$ be open affine neighborhoods of $\xi$ and $\phi(\xi)$ in $X_1$ and $X_2$, respectively, such that $\phi(V_{1,k}) \subseteq V_2$. Write $V_1 = \operatorname{Spec} A_1$ and $V_2 = \operatorname{Spec} A_2$. Then $\phi$ determines a $k$-algebra homomorphism

$$\alpha\colon A_2 \otimes_B k \to A_1 \otimes_B k\,.$$

By flatness of $A_1$ over $B$, the map $A_1 \to A_1 \otimes_B k$ is injective, so we may view $A_1$ as a subring of $A_1 \otimes_B k$. Let $x_1, \dots, x_n$ be a system of generators for $A_2$ over $B$. Then, for each $i$, there is a nonzero $f_i \in k$ such that $f_i \alpha(x_i \otimes 1) \in A_1$. Let $f = f_1 \dots f_n$; then the map $x \mapsto \alpha(x \otimes 1)$ extends to a $B$-algebra homomorphism $\beta\colon A_2 \to (A_1)_f$, which is unique by flatness. Thus the restriction of $\phi$ to the generic fiber of $V_1$ extends uniquely to a morphism $D(f) \to V_2$. This proves the claim.

Letting $\xi$ vary, by the uniqueness assertion the various extensions patch together to give a Zariski-open subset $V \subseteq X_1$ containing $X_{1,k}$, such that $\phi$ extends to a morphism $\psi\colon V \to X_2$ over $Y$. Since $V$ contains the generic fiber, all irreducible components of its complement lie over proper Zariski-closed subsets of $Y$; removing those subsets then gives the desired open subset $U$ of $Y$. □

**Corollary 2.14.** *Let $X$ be a scheme of finite type over a noetherian integral scheme $Y$. Then, for any regular function $f$ on the generic fiber of $X \to Y$, there is a nonempty open subset $U \subseteq Y$ such that $f$ extends to a regular function on $X \times_Y U$.*

*Proof.* Apply Lemma 2.13 to the morphism $X \times_Y \operatorname{Spec} K(Y) \to \mathbb{A}^1_{K(Y)}$ associated to $f$. □

**Proposition 2.15.** *Let $W$ be a scheme of finite type over a global field $k$, let $Y$ be a model for $k$, and let $X_1$ and $X_2$ be models for $W$ over $Y$. Then there is a nonempty open subscheme $Y'$ of $Y$ such that $X_1 \times_Y Y' \cong X_2 \times_Y Y'$; i.e., they differ at only finitely many places. Likewise, if a coherent sheaf $\mathscr{F}$ on $W$ extends to coherent sheaves $\mathscr{F}_1$ on $X_1$ and $\mathscr{F}_2$ on $X_2$, then $Y'$ can be chosen such that $\mathscr{F}_1$ is taken to $\mathscr{F}_2$ by the above isomorphism. A similar statement holds for Cartier divisors.*

*Proof.* By Lemma 2.13, there is a nonempty open subscheme $Y'$ of $Y$ such that, letting $X_1' = X_1 \times_Y Y'$ and $X_2' = X_2 \times_Y Y'$, the isomorphism between the generic fibers of $X_1$ and $X_2$ extends to a morphism $\phi\colon X_1' \to X_2'$, and the inverse of this isomorphism extends to a morphism $\psi\colon X_2' \to X_1'$. The compositions $\psi \circ \phi$ and $\phi \circ \psi$ agree with the identity morphisms on $X_1'$ and $X_2'$, respectively, on the generic fibers, hence everywhere by Propositions **0:**3.11 and **0:**3.12. Thus $\phi$ and $\psi$ are mutually inverse, proving the first assertion.

The assertions regarding coherent sheaves and Cartier divisors are left to the reader as exercises. □

The following proposition allows us to choose the closed fibers on models independently (up to the restriction indicated by Proposition 2.15).

**Proposition 2.16.** *Let $W$ be a scheme over a global field $k$, let $Y$ be a model for $k$, let $X$ and $X'$ be models for $W$ over $Y$, and let $v$ be a non-archimedean place of $k$ corresponding to a closed point $v \in Y$. Then there exists a model $X''$ for $W$ over $Y$ such that*

$$X'' \times_Y (Y \setminus \{v\}) \cong X \times_Y (Y \setminus \{v\})$$

*and*

$$X'' \times_Y \operatorname{Spec} \mathscr{O}_{Y,v} \cong X' \times_Y \operatorname{Spec} \mathscr{O}_{Y,v} .$$

*If $X$ and $X'$ are proper, then so is $X''$. Finally, if $\mathscr{F}$ and $\mathscr{F}'$ are coherent sheaves on $X$ and $X'$, respectively, agreeing on $W$, then there is a coherent sheaf $\mathscr{F}''$ on $X''$ compatible with $\mathscr{F}$ and $\mathscr{F}'$ under the above isomorphisms; moreover, if $\mathscr{F}$ and $\mathscr{F}'$ are vector sheaves, then so is $\mathscr{F}''$. A similar assertion holds for Cartier divisors.*

*Proof.* Let $Y'$ be a nonempty open subset of $Y$ such that $X$ and $X'$ agree over $Y'$. We may assume that $v \notin Y'$. Then we may glue the schemes $X \times_Y (Y \setminus \{v\})$ and $X' \times_Y (Y' \cup \{v\})$ along their isomorphic parts over $Y'$ to get a scheme $X''$ over $Y$. By construction, it is a model for $W$ over $Y$, and has the desired properties. The assertions regarding coherent sheaves and Cartier divisors can be proved by choosing $Y'$ such that the respective sheaves or divisors are compatible over $Y'$. □

**Remark 2.17.** By Lemma 2.8 or Theorem 2.12, the above proposition remains true when $X'$ is replaced by a local model for $W$ over $Y_v$.

**Remark 2.18.** In Proposition 2.16, if $X$ and $X'$ are projective models, then it is not clear whether $X''$ is also a projective model. The reason for this is that a relatively ample line sheaf on $X$ might not be relatively ample on $X'$. In fact, it might not even extend as a line sheaf.

## §3. Models and points on the corresponding variety

The whole point of models is to be able to work with rational and algebraic points on a variety $X_k$ in terms of curves on a corresponding model $X$. One can then apply algebraic geometry to obtain properties of those curves.

This section sets up the correspondence between points on $X_k$ and curves on $X$.

*Throughout this section, $k$ is a global or non-archimedean local field, $Y$ is a model or local model for $k$, $\pi\colon X \to Y$ is a scheme of finite type, and $X_k$ is the generic fiber of $\pi$. (If $\pi$ is flat then $X$ is a model for $X_k$, but flatness is not needed in this section.)*

**Proposition 3.1.** *Let $L$ be a finite extension of $k$, and let $Y' \to Y$ be the normalization of $Y$ in $L$. Then $Y'$ is a model or local model for $L$, and it is a proper model for the variety $\operatorname{Spec} L$ over $Y$. Moreover, there is a natural one-to-one correspondence between the set of all rational maps $Y' \dashrightarrow X$ over $Y$ and $X_k(L)$, given by restricting a rational map to the generic point of $Y'$.*

$$\begin{tikzcd} & X \arrow[d, "\pi"] \\ Y' \arrow[ur, dotted] \arrow[r] & Y \end{tikzcd}$$

*Proof.* The first assertion is easy to check, by reference to each part of Definition 1.1 separately, or to Definition 1.2 when $k$ is a local field. By construction $Y'$ is integral; hence it is a model for $\operatorname{Spec} L$ by Remark 1.4, and it is a proper model since it is finite over $Y$.

The inverse of the map from rational maps $Y' \dashrightarrow X$ to $X_k(L)$ is given by Lemma 2.13, and the provisions of that lemma imply that this map is indeed the inverse.

This bijection is functorial in the following sense. Let $X'$ be a scheme of finite type over $Y$, let $f\colon X \to X'$ be a morphism over $Y$, let $j\colon Y' \dashrightarrow X$ be a rational map, and let $j_L \in X_k(L)$ be the corresponding map on generic fibers. Then the rational map $f \circ j\colon Y' \dashrightarrow X'$ corresponds to $f(j_L) \in X'_k(L)$. $\square$

**Remark 3.2.** Since $X_k(L) = X(L)$, we will often use the latter notation from now on, since it is shorter.

**Corollary 3.3.** *The natural map from the set of rational sections $\sigma\colon Y \dashrightarrow X$ to $X(k)$, given by restricting to the generic fiber, is bijective.*

*Proof.* Indeed, in this case $Y' = Y$. (Recall that a rational section is a rational map $\sigma\colon Y \dashrightarrow X$ such that the rational map $\pi \circ \sigma$ coincides with the identity on $Y$.) □

**Proposition 3.4.** *If $X$ is a proper model, then all of the rational maps $Y' \dashrightarrow X$ in Proposition 3.1 and Corollary 3.3 extend to morphisms; thus $X(Y') \to X_k(L)$ is bijective.*

*Proof.* Use the valuative criterion of properness. □

Instead of working with morphisms, one can work instead with closed points on $X_k$ and closed curves in $X$, as follows.

**Definition 3.5.** An integral subscheme $Z$ of $X$ is **vertical** if it is contained in a closed fiber of $\pi$, and it is **horizontal** if it dominates $Y$ via $\pi$ (*i.e.*, if it is not vertical).

Recall also that a curve in $X$ is a closed (integral) subvariety of dimension $1$.

**Proposition 3.6.** *Assume that $X$ is proper over $Y$. Then there is a natural one-to-one correspondence between the sets of closed horizontal curves $Z$ in $X$ and closed points $x \in X_k$, given by $Z \mapsto x$ where $Z \cap X_k = \{x\}$. The inverse of this map is $x \mapsto \overline{\{x\}}$.*

*Proof.* First of all, $Z \cap X_k$ is a closed subset of $X_k$; it is nonempty since $Z$ is horizontal, it is of dimension zero because it is a proper subset of the irreducible set $Z$ of dimension $1$, and it consists of only one point because otherwise $Z$ would not be irreducible. On the other hand, $\overline{\{x\}}$ is a closed subset of $X$; it is irreducible because it is the closure of a single point; it is horizontal because it meets the generic fiber; and it has dimension $1$ because fibers over $Y$ have dimension $0$, and $\dim Y = 1$.

Let $\phi$ and $\psi$ denote the maps $Z \mapsto x$ where $Z \cap X_k = \{x\}$ and $x \mapsto \overline{\{x\}}$, respectively. Let $x \in X_k$ be a closed point. Then the set $\psi(x) \cap X_k$ is a subset of $X_k$ containing $x$, yet it contains exactly one point by the above paragraph. Thus $\phi \circ \psi$ is the identity on the set of closed points of $X_k$. On the other hand, let $Z$ be a closed integral curve in $X$. Then $\psi(\phi(Z))$ is obtained by intersecting with the generic fiber, and then taking the closure. This gives a nonempty closed subset of $Z$ of dimension $1$, which must be all of $Z$ since the dimensions are the same and $Z$ is irreducible. Thus $\phi$ and $\psi$ are mutually inverse.

Functoriality of this bijection follows as in the proof of Proposition 3.1. □

**Proposition 3.7.** *Under the above correspondence, rational points $x \in X_k$ (i.e., closed points $x \in X_k$ with $k(x) = k$) correspond to closed images of rational sections $\sigma\colon Y \dashrightarrow X$ of $\pi$.*

*Proof.* First, let $x$ be a rational point. The isomorphism $k(x) = k$ gives an element of $X(k)$, which by Corollary 3.3 gives a rational section $\sigma\colon Y \dashrightarrow X$. This map, from rational points to rational sections, is bijective. The image of $\sigma$ is contained in the

closed curve $Z$ corresponding to $x$. As in the proof of Proposition 3.6, the closure of the image of $\sigma$ contains $x$, hence it equals $Z$.

Now suppose $z \in Z$ does not lie in the image of $\sigma$. Then $y := \pi(z)$ is a closed point in $Y$. Via the isomorphism $K(Y) \xrightarrow{\sim} K(Z)$ induced by $\pi$, the local ring $\mathscr{O}_{Z,z}$ dominates the local ring $\mathscr{O}_{Y,y}$. Since the latter is a valuation ring, the two local rings are equal; hence no other point of $Z$ maps to $y$, and by looking at affine neighborhoods of these two points it can be shown that $\sigma$ extends to a morphism mapping $y$ to $z$. Thus, if $U$ is the maximal domain of $\sigma$, then its image is all of $Z$.

Finally, no two distinct rational maps $\sigma_1$, $\sigma_2$ can have the same image $Z$, because they would have to come from the same point in $X(k)$. □

Models are also a very convenient tool for working with integral points, because the definition of integral point using models is more specific than the earlier definition using Weil functions.

**Definition 3.8.** Let $k$, $S$, and $Y$ be as in Section 7 of Chapter 2, allowing also $S = \emptyset$ when $k$ is a function field. Note that $Y$ is a model for $k$. Let $X$ and $X_k$ be as defined at the beginning of this section. Let $L$ be an algebraic extension of $k$, and let $Y'$ be the integral closure of $Y$ in $L$. Then we say that a point $P \in X(L)$ is **integral over** $Y'$ (or integral over $\mathscr{O}(Y')$, if $Y'$ is affine) if the rational map $Y' \dashrightarrow X$ associated to $P$ by Proposition 3.1 extends to a morphism $Y' \to X$.

**Remark 3.9.** The case $L = k$ is the most important case. In that case the set of integral points is just $X(Y)$.

**Remark 3.10.** When $X$ is proper over $Y$, the integrality condition is vacuous, by the valuative criterion of properness (Proposition 3.4). Instead of disregarding the condition of integrality in the proper case, though, it is often more useful to think of integrality as a generalization of the idea of being a rational or algebraic point.

**Remark 3.11.** If $X_1$ and $X_2$ are schemes of finite type over $Y$, and if $f\colon X_1 \to X_2$ is a morphism over $Y$, then for any integral point $P$ on $X_1$, its image $f(P)$ is an integral point on $X_2$. This corresponds to Proposition **2:**7.7. In this case, though, it is immediate from the definition.

**Remark 3.12.** As was the case when defining integrality using Weil functions (Definition **2:**7.4), it is often better to phrase the notion of integrality in terms of a proper scheme (or model) minus a divisor. To do this in this case, embed $X$ into a proper scheme $\overline{X}$ over $Y$. (One can embed $X$ as a dense open subscheme of a proper scheme $\overline{X}$ by Nagata's embedding theorem (Theorem **B:**4.1); if $X$ is flat over $Y$ then so is $\overline{X}$ by Propositions **0:**3.2 and **0:**3.11.) By blowing up, we may assume that the closed subset $\overline{X} \setminus X$ is the support of an effective divisor $D$ on $\overline{X}$. One then has the notion of $D$-integrality for rational or algebraic points corresponding to morphisms $Y' \to \overline{X}$: such a point is $D$-integral if and only if the image of $Y' \to \overline{X}$ does not meet the support of $D$. Moreover, we will see later (Remark 5.13) that this definition coincides with the definition of integral point via Weil functions (Definition **2:**7.4).

## §4. An example

This section gives an example showing the use of algebraic geometry to prove a diophantine result in the function field case.

**Theorem 4.1** (Fermat's Last Theorem for function fields). *Let $F$ be a field of characteristic $0$, and let $n \geq 3$ be an integer. Then the only relatively prime polynomials $a(t), b(t), c(t) \in F[t]$ satisfying the equation $a^n + b^n + c^n = 0$ are constants.*

*Proof.* We may assume that $F$ is algebraically closed.

Let $k = F(t)$, and let $X_k$ be the curve $x^n + y^n + z^n = 0$ in $\mathbb{P}^2_k$. Also let $Y = \mathbb{P}^1_F$, and let $X_0$ be the curve $x^n + y^n + z^n = 0$ in $\mathbb{P}^2_F$. Then $Y$ is a model for $k$, and $X := X_0 \times_F \mathbb{P}^1_F$ is a model for $X_k$, with projection $\pi\colon X \to Y$ given by projection to the second factor. A triple $(a, b, c)$ of polynomials as above gives a point $x \in X(k)$, and hence a section $\sigma\colon Y \to X$ of $\pi$, by Proposition 3.4.

Let $q\colon X \to X_0$ be the projection to the first factor. Then $q \circ \sigma$ gives a morphism $Y = \mathbb{P}^1_F \to X_0$. This morphism is characterized by the condition that if $y \in Y$ is a closed point corresponding to a number $t \in F = \mathbb{A}^1_F \subseteq \mathbb{P}^1_F$, then $q(\sigma(y))$ is the point in $X_0(F)$ with homogeneous coordinates $[a(t) : b(t) : c(t)]$.

Assume now that not all of $a, b, c$ are constant. Then some ratio $a/b$ or $a/c$ is not constant; hence $q \circ \sigma$ is not a constant map. Therefore it is a finite map. However, $Y$ is a curve of genus $0$, and since $n \geq 3$, $X_0$ is a curve of genus $g > 0$. This contradicts Lüroth's Theorem ([**H**], IV Example 2.5.5). Thus $a$, $b$, and $c$ must be constant. □

**Remark.** The fact that $X$ could be descended to a curve over the field of constants $F$, allowing us to use a product variety as a model, is called the **split function field case**. Working in the split function field case is considerably easier than working in the general function field case (which, in turn, is usually easier than working in the number field case). Of course, for the above theorem, the lack of a "sideways" morphism $q$ contributes to the fact that Wiles's proof of the number field case of Fermat's Last Theorem runs for hundreds of pages.

## §5. Models and Weil functions

The theory of models allows us to fill a gap left over in Chapter 2, namely, the existence of Weil functions on general proper schemes over $k$.

The following discussion involves functions which would be Weil functions, except that they only exist over those places of the given local or global field $k$ represented by closed points in a given model for $k$.

**Definition 5.1.** Let $k$ be a global or non-archimedean local field, let $Y$ be a model for $k$, and let $W$ be a scheme of finite type over $k$.

(a). Then

$$W(M_Y) = \coprod W(\mathbb{C}_v) ,$$

where the disjoint union is over the set of places $v$ of $k$ corresponding to closed points of $Y$. If $W$ is proper over $k$ and if $X$ is a proper model for $W$ over $Y$, then we also write $X(M_Y) = W(M_Y)$.

(b). A function $f\colon W(M_Y) \to \overline{\mathbb{R}}$ is said to be **locally $M_Y$-bounded from below** (resp. **locally $M_Y$-bounded from above**) if it is bounded from below (resp. from above) by an $M_k$-constant on all $M$-bounded subsets of $W(M_Y)$, and $f$ is **$M_Y$-bounded** if it is $M_Y$-bounded from below and from above.

Now let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $X$ be a proper scheme over $Y$, and let $D$ be a Cartier divisor on $X$. Let $v$ be a non-archimedean place of $k$ corresponding to a closed point in $Y$, and let $R_v$ be the valuation ring of $\mathbb{C}_v$. Then, by the valuative criterion of properness, $X(\mathbb{C}_v) = X(R_v)$. Corresponding to a point $P \in X(\mathbb{C}_v)$, there is therefore a morphism $\phi_P\colon \operatorname{Spec} R_v \to X$ over $Y$. Assume now that $P \notin \operatorname{Supp} D$. Then the image of the generic point of $\operatorname{Spec} R_v$ via $\phi_P$ does not lie in $\operatorname{Supp} D$, so the pull-back $\phi_P^* D$ exists as a Cartier divisor on $\operatorname{Spec} R_v$.

This leads to the question of what the group of Cartier divisors on $\operatorname{Spec} R_v$ looks like. Since $R_v$ is not noetherian, this group does not look at all like the group of Weil divisors. Instead, we note that $R_v$ is entire, so the sheaves $\mathscr{K}^*$ and $\mathscr{O}^*$ on $\operatorname{Spec} R_v$ are just the constant sheaves $\mathbb{C}_v^*$ and $R_v^*$, respectively. Therefore the group of Cartier divisors is (canonically) isomorphic to $\mathbb{C}_v^*/R_v^*$, which in turn is isomorphic to the valuation group, which is $\mathbb{Q}$.

Since the group of Cartier divisors on $\operatorname{Spec} R_v$ is closely related to the valuation on $\mathbb{C}_v$, we can define a norm $\|\cdot\|_v$ on this group as follows.

**Definition 5.2.** Let $E$ be a Cartier divisor on $\operatorname{Spec} R_v$, represented by a function $x \in \mathbb{C}_v^*$ in the (unique) open neighborhood of the closed point. This function $x$ is unique only up to multiplication by an element of $R_v^*$, but its norm $\|x\|$ is independent of the choice of $x$, and we define $\|E\|_v = \|x\|$. (The subscript $v$ reappears here because the scheme $\operatorname{Spec} R_v$ lacks information about the normalization of the absolute value on $\mathbb{C}_v$.) If $L$ is a finite extension of $k$ and $w$ is a place of $L$ with $w \mid v$, then we also write $\|E\|_w = \|E\|_v^{[L_w:k_v]}$.

This definition leads to the following definition of a function which in many respects is like a Weil function, but it is not defined at any of the archimedean places of $k$. Thus, we may say that it is a **partial Weil function**.

**Definition 5.3.** With notation as above, we define a function

$$\lambda_D\colon (X \setminus \operatorname{Supp} D)(M_Y) \to \mathbb{R}$$

by

$$\lambda_D(P) = -\log \|\phi_P^* D\|_{v(P)} ,$$

where $v(P)$ is the place for which $P \in W(\mathbb{C}_v)$.

The notation here is the same as for a Weil function in Chapter 2; what distinguishes the two definitions is that here $D$ is a divisor on $X$, whereas in the earlier definition $D$ was a divisor on $X_k$.

**Lemma 5.4.** *This function $\lambda_D$ has the following properties.*

(a). *It is additive: if $D_1$ and $D_2$ are Cartier divisors on $X$, then*

$$\lambda_{D_1+D_2}(P) = \lambda_{D_1}(P) + \lambda_{D_2}(P)$$

*for all $P \in (X \setminus (\operatorname{Supp} D_1 \cup \operatorname{Supp} D_2))(M_Y)$;*

(b). *It is functorial in $D$;*

(c). *If $D$ is effective then $\lambda_D(P) \geq 0$ for all $P$; and*

(d). *If $D$ is principal, say $D = (f)$, then $\lambda_D(P) = -\log \|f(P)\|_{v(P)}$ for all $P$.*

*Proof.* These are trivial verifications. For example, if $D$ is effective, then so is $\phi_P^* D$; therefore $\phi_P^* D$ is represented by an element of $R_v$, so $\|\phi_P^* D\|_v \leq 1$. □

**Remark 5.5.** Note that the above equations and inequalities hold *exactly,* not just up to $O(1)$. It is also possible to define Weil functions (Definition 5.3) and heights (6.6) using models; when defined that way they can be treated exactly, instead of up to $O(1)$. This is an advantage of working with models; one can think of choosing a model and an extension of a given divisor to that model, as being equivalent to fixing a Weil function for that divisor (at least at non-archimedean places). Subsequent chapters will show how fixing an arithmetic scheme and an extension of a divisor as an arithmetic divisor on the arithmetic scheme does the same for the whole Weil function.

Before investigating the properties of the functions $\lambda_D$, a better understanding of the structure of the total quotient ring (Definition **0:**4.1) is needed.

**Lemma 5.6.**

(a). *Let $A$ be a reduced noetherian ring, and let $S$ be the multiplicative system of nonzero elements which are not zero divisors. Then the total quotient ring $S^{-1}A$ is the product of the fields of fractions of the affine rings of the irreducible components of* $\operatorname{Spec} A$.

(b). *Let $X$ be a reduced noetherian scheme. Then the sheaf $\mathscr{K}$ of total quotient rings of $X$ is the product of the skyscraper sheaves $K(X_i)$ at the generic points of all the irreducible components $X_i$ of $X$.*

(c). *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $X$ be a $Y$-scheme of finite type, and let $i\colon X_k \to X$ be the inclusion map. Then $i_*\mathscr{K}_{X_k} \cong \mathscr{K}_X$.*

*Proof.* Let $A_1, \dots, A_n$ be the affine rings of the irreducible components $X_1, \dots, X_n$ of $\operatorname{Spec} A$. Since $\operatorname{Spec}(A_1 \times \cdots \times A_n)$ is the disjoint union of the $X_i$, which maps surjectively onto $\operatorname{Spec} A$, the natural map

$$A \to A_1 \times \cdots \times A_n$$

is injective. This defines an injective map from $A$ to the product of the fields of fractions $K_i$ of the $A_i$, which in turn defines a map

$$\phi\colon S^{-1}A \to \prod K_i\ ,$$

which is also injective.

Moreover, for all $i$, the kernel of the map $A \to A_1 \times \cdots \times \widehat{A}_i \times \cdots \times A_n$ maps injectively to $A_i$; let $\mathfrak{a}_i$ be its image. (This is the ideal corresponding to the sheaf intersection of $X_i$ with the union of the other irreducible components. Indeed, think of the intersection as the base change of one closed immersion by the other, and translate back into the language of rings.) If $f_i \in A_i$ lie in $\mathfrak{a}_i$ for all $i$, then $(f_1, \dots, f_n) \in A$. This fact can be used to show surjectivity of $\phi$, as follows. Pick an element $(a_1/s_1, \dots, a_n/s_n) \in \prod K_i$ with $a_i, s_i \in A_i$ and $s_i \neq 0$ for all $i$. After multiplying both $a_i$ and $s_i$ by a nonzero element of $\mathfrak{a}_i$ if necessary, we may assume that $a_i, s_i \in \mathfrak{a}_i$. Then $(a_1, \dots, a_n) \in A$ and $(s_1, \dots, s_n) \in S$, so the chosen element lies in the image of $\phi$. Thus part (a) holds.

Part (b) is the translation of (a) into the language of sheaves and schemes, and part (c) is immediate since the generic points of $X_k$ and $X$ are the same, and the corresponding irreducible components have the same function fields. □

The main goal of this section is to show that $\lambda_D$ satisfies many of the conditions of a Weil function. We start with continuity.

**Lemma 5.7.** *The function $\lambda_D$ is continuous (in the $v$-topology).*

*Proof.* It will suffice to consider just one non-archimedean place $v$.

Let $U = \operatorname{Spec} A$ be an open affine subset of $X$ on which $D$ is locally represented by a function $f$ in the total quotient ring $S^{-1}A$. Let $P \in U(R_v)$ be a point such that $\phi_P$ does not take $\operatorname{Spec} \mathbb{C}_v$ to a point in $\operatorname{Supp} D$. Then $\lambda_D(P) = -\log \|f(P)\|_v$, which is continuous in $P$ by definition of the $v$-topology.

Note that the set of $P$ as above is strictly smaller than $(U \setminus \operatorname{Supp} D)(\mathbb{C}_v)$, but as the sets $U$ vary over an open cover of $X$, the sets considered above give an open cover of $(X \setminus \operatorname{Supp} D)(\mathbb{C}_v)$ in the $v$-topology. □

**Lemma 5.8.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $X$ be a proper scheme over $Y$, let $D$ be an effective Cartier divisor on $X$, and let $U$ be an open affine subset of $X_k$ disjoint from $\operatorname{Supp} D$. Then $\lambda_D$ is locally $M_Y$-bounded from above on $U(M_Y)$.*

*Proof.* First consider the special case in which $X = \mathbb{P}^n_Y$, $U = \mathbb{A}^n_k$, and $D$ is the hyperplane at infinity. Write $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$, and identify $\mathbb{A}^n_k$ with the subset $D_+(x_0)$ in the usual way, so that $D$ is given by $\{x_0 = 0\}$, both on $X_k$ and on $X$. Let $v$ be a place of $k$ corresponding to a closed point of $Y$, let $P \in \mathbb{P}^n(\mathbb{C}_v)$ have homogeneous coordinates $[\xi_0 : \cdots : \xi_n]$ in $\mathbb{C}_v$, and pick $i \in \{0, \dots, n\}$ such that $\|\xi_i\|$ is maximal.

Then the image of $\phi_P\colon \operatorname{Spec} R_v \to X$ is contained in the open affine $x_i \neq 0$, on which $D$ is defined by the rational function $x_0/x_i$. (This holds even if $i = 0$.) The divisor $\phi_P^* D$ is the principal divisor $(\xi_0/\xi_i)$, so $\lambda_D(P) = -\log \|\xi_0/\xi_i\|$. For $j = 0, \dots, n$ let $y_j = x_j/x_0$; then $y_0, \dots, y_n$ generate the affine ring $\mathscr{O}(U)$ over $k$, and if

$$P \in B(U, y_0, \dots, y_n, \gamma)$$

for some $M_k$-constant $\gamma$, then

$$\lambda_D(P) = \log \|y_i(P)\| \leq \gamma_v \ .$$

This shows that $\lambda_D$ is locally $M_Y$-bounded from above in this case.

Now consider the case in which $X$ is integral. Let $y_1, \dots, y_n$ be a system of generators for $\mathscr{O}(U)$ over $k$; this choice defines a closed immersion $j\colon U \hookrightarrow \mathbb{A}^n_k$. This in turn determines a rational map $X \dashrightarrow \mathbb{P}^n_Y$; let $Z$ be the normalization of the closure of the graph of this rational map, and let $p\colon Z \to X$ and $q\colon Z \to \mathbb{P}^n_Y$ be the projection morphisms. Also let $H$ denote the hyperplane at infinity on $\mathbb{P}^n_Y$. By Proposition **0:**3.9 and the assumption on the support of $D$, we have $\operatorname{Supp} p^* D \subseteq \operatorname{Supp} q^* H$ on the generic fiber. Since $Z$ is normal, Cartier divisors on $Z$ can be viewed also as Weil divisors, and we may therefore find an integer $m$ such that $mq^*H - p^*D$ is effective on the generic fiber (as a Weil divisor). Let $\pi\colon Z \to Y$ be the structural morphism; then there is a divisor $E$ on $Y$ such that $mq^*H - p^*D + \pi^*E$ is effective on all of $Z$. By ([**H**], II Prop. 6.3A) it is therefore effective as a Cartier divisor. By Lemmas 5.4a–5.4c and the fact that $p$ induces a surjection $Z(M_Y) \to X(M_Y)$, we then have, for an $M_k$-constant $\gamma$ (assocated to $E$),

$$\lambda_D(P) \leq m\lambda_H(j(P)) + \gamma$$

for all $P \in U(M_Y)$. This reduces the integral case to the case of $\mathbb{P}^n$ proved above.

To prove the general case, we immediately reduce to the case where $X$ is reduced. The result then follows by applying the integral case to the restriction of $\lambda_D$ to each irreducible component, and then applying (the counterpart for $M_Y$-bounded functions of) Proposition **2:**3.2g. □

These lemmas combine to prove the following theorem, which is a major step in the main result of the section (existence of Weil functions).

**Theorem 5.9.** *Let $k$ be a global or non-archimedean local field, let $Y$ be a model or local model for $k$, let $X$ be a proper scheme over $Y$, let $D$ be a Cartier divisor on $X$, and let $\lambda_D\colon (X \setminus \operatorname{Supp} D)(M_Y) \to \mathbb{R}$ be as in Definition 5.3. Then, for all open affines $U = \operatorname{Spec} A$ in $X_k$ such that the restriction of $D$ to $U$ is given by a principal divisor $(f)$, there exists a continuous locally $M_Y$-bounded function $\alpha\colon U(M_Y) \to \mathbb{R}$ such that*

$$\lambda_D(P) = -\log \|f(P)\|_{v(P)} + \alpha(P) \tag{5.9.1}$$

*for all* $P \in (U \setminus \operatorname{Supp} D)(M_Y)$ .

*Proof.* By parts (b) and (c) of Lemma 5.6, the section $f$ of $\mathscr{K}_{X_k}$ over $U$ extends to a global section of $\mathscr{K}_X$ , which we also denote by $f$ . Let $D' = D - (f)$ . Then $\operatorname{Supp} D'$ is disjoint from $U$ , and $\lambda_{D'}$ (defined by Definition 5.3) is defined on all of $U(M_Y)$ . Moreover, by parts (a) and (d) of Lemma 5.4, $\alpha := \lambda_{D'}$ satisfies (5.9.1).

By Lemma 5.7, $\alpha$ is continuous. By Lemma 5.8, applied to $D'$ and to $-D'$ , $\alpha$ is locally $M_Y$-bounded. □

At archimedean places, existence of Weil functions is much easier.

**Lemma 5.10.** *Let $k$ be a global or archimedean local field, let $W$ be a proper scheme over $k$ , let $D$ be a Cartier divisor on $W$ , and let $v$ be an archimedean place of $k$ . Then there is a local Weil function $\lambda_{D,v}$ for $D$ at $v$ .*

*Proof.* Let $\{(U_i, f_i)\}$ be a system of representatives for $D$ . We may assume that it is a finite collection. Let $\{\phi_i \colon W(\mathbb{C}_v) \to \mathbb{R}\}$ be a continuous partition of unity on $W(\mathbb{C}_v)$ , with $\operatorname{Supp} \phi_i \Subset U_i(\mathbb{C}_v)$ for all $i$ . (Here $\operatorname{Supp} \phi_i$ means the set where $\phi_i$ is nonzero, not the support as defined for Weil functions.) Then

$$\sum_i \phi_i \cdot (-\log \|f_i\|)$$

is a local Weil function for $D$ at $v$ . □

This leads to the main theorem of the section.

**Theorem 5.11.** *Let $k$ be a local or global field, let $W$ be a proper scheme over $k$ , and let $D$ be a Cartier divisor on $W$ . Then there exists a Weil function for $D$ .*

*Proof.* By Lemma 5.10, there exist local Weil functions $\lambda_{D,v}$ for all archimedean $v \in M_k$. If $k$ is an archimedean local field, then we are done. Otherwise, let $Y$ be a maximal model for $k$ ; let $X$ be a proper model for $W$ over $Y$ , on which $D$ extends to a Cartier divisor $D'$ ; and let $\lambda_{D'}$ be as in Definition 5.3. Then

$$\lambda_D(P) = \begin{cases} \lambda_{D,v(P)}(P), & \text{if } v(P) \text{ is archimedean;} \\ \lambda_{D'}(P), & \text{otherwise} \end{cases}$$

defines a Weil function $\lambda_D$ , as desired, by Theorem 5.9 and an argument similar to Proposition **2:**5.9. □

**Remark 5.12.** Given a model $X$ for $W$ over a model or local model for $k$ and an extension of $D$ to a Cartier divisor on $X$ , the above Weil function can be chosen to agree with the partial Weil function of Definition 5.3 associated to this divisor (on the domain of the partial Weil function).

Definition 5.3 and Theorem 5.9 allow for a comparison of the present definition of integral point using models, with the earlier definition in terms of Weil functions.

**Remark 5.13.** Let $k$, $Y$, $X$, and $X_k$ be as in Section 3, with $X$ proper over $Y$, and let $D$ be an effective Cartier divisor on $X$. Choose a Weil function $\lambda_D$ on $X_k$ with respect to $D\big|_{X_k}$ that extends the partial Weil function of Definition 5.3. Let $L$ be any finite extension of $k$, let $Y'$ be the normalization of $Y$ in $L$, let $P$ be an element of $X(L)$, and let $\phi_P\colon Y' \to X$ be the corresponding morphism. Then the image of $\phi_P$ is disjoint from $\operatorname{Supp} D$ if and only if $\lambda_{D,w}(P) \le 0$ for all places $w \in M_L$ lying over places represented by closed points in $Y$. In other words, $P$ is integral with respect to this choice of model (by Definition 3.8) if and only if it is integral with respect to this choice of Weil function (by Definition **2:**7.4).

An approximate converse of this statement also holds.

**Lemma 5.14.** *Let $k$, $Y$, $X$, and $X_k$ be as in Section 3, with $X$ proper over $Y$, let $D$ be an effective Cartier divisor on $X$, let $\lambda_D$ be the partial Weil function of Definition 5.3, and let $\gamma$ be an $M_k$-constant. Then there exists another model $X'$ for $X_k$, isomorphic to $X$ at all non-archimedean places $v$ of $k$ for which $\gamma_v \le 0$, and an effective Cartier divisor $D'$ on $X'$, compatible with $D$ at all $v$ as above, such that if $\lambda'$ is the partial Weil function associated to $D'$, then $\lambda' \le \max\{0, \lambda_D - \gamma\}$ (where we identify $X(M_Y)$ with $X'(M_Y)$).*

*Proof.* Let $E$ be an effective divisor on $Y$, such that if $y \in k^*$ represents $E$ locally near a point of $Y$ corresponding to a place $v$, then $-\log \|y\|_v \ge \gamma_v$. We also assume that $E$ is not supported at any points of $Y$ corresponding to places $v$ for which $\gamma_v \le 0$. Let $\mathscr{I}_E$ be the sheaf of ideals corresponding to the pull-back of $E$ to $X$, let $\mathscr{I}_D$ be the sheaf of ideals corresponding to $D$, and let $\pi\colon X' \to X$ be the blowing-up of $X$ along the sheaf of ideals $\mathscr{I} := \mathscr{I}_D + \mathscr{I}_E$. Then $\pi$ is an isomorphism on the complement of the closed subset defined by $\mathscr{I}_E$. On open affines $U = \operatorname{Spec} A$ of $X$ on which $D$ and $E$ are principal divisors $(f)$ and $(g)$, respectively, then $\pi^{-1}(U)$ is covered by open affines $D_+(f) = \operatorname{Spec} A[g/f]$ and $D_+(g) = \operatorname{Spec} A[f/g]$, and we let $D'$ be the Cartier divisor represented on $D_+(f)$ by $(1)$ and on $D_+(g)$ by $(f/g)$. This is well defined because $X'$ can be viewed as a closed subscheme of $\mathbb{P}^1_A$ via the graded surjection $A[x,y] \to \bigoplus \mathscr{I}^d$ given by $x \mapsto f$, $y \mapsto g$, and $D'$ is the restriction of the divisor $(x)$ there. This divisor is invariant under multiplying $x$ or $y$ by units of $A$, and coincides with $D$ away from the closed subset defined by $\mathscr{I}_E$.

It remains only to check the assertion regarding $\lambda'$. Let $L$ be a finite extension of $k$, let $w$ be a place of $L$ lying over a non-archimedean place $v \in M_k$, let $R_w$ be the valuation ring of $L_w$, and let $\phi\colon \operatorname{Spec} R_w \to X$ be the map corresponding to a point in $P \in X(L)$ not lying in the support of $D$. Then $\phi$ lifts to $\phi'\colon \operatorname{Spec} R_w \to X'$. Let $U = \operatorname{Spec} A$, $f$, and $g$ be as above, such that $U$ contains the image under $\phi$ of the closed point of $\operatorname{Spec} R_w$. Then $\phi'$ takes that closed point to a point of $D_+(f)$ or $D_+(g)$. In the former case, $(\phi')^*D'$ vanishes, and so $\lambda'_w(P) = 0$. In the latter case, $(\phi')^*D' = (f/g)$, so $\lambda'_w(P) = \lambda_w(P) + \log \|g\|_w \le \lambda_w(P) - \gamma_w$, where we have assumed that $g \in k^*$. $\square$

**Theorem 5.15.** *Let $k$ be a global or non-archimedean local field, let $W$ be a proper scheme over $k$, let $D$ be an effective Cartier divisor on $W$, let $\lambda_D$ be a Weil*

*function for $D$, and let $Y$ be a model or local model for $k$. Then there is a proper model $X$ for $W$ over $Y$ and an extension of $D$ to a Cartier divisor $E$ on $X$ such that if $\lambda'$ denotes the partial Weil function associated to $E$ as in Definition 5.3, then $\lambda' \le \max\{0, \lambda_D\}$. In particular, all $\lambda_D$-integral points on $W$ correspond to integral points on $X$.*

*Proof.* Let $X_0$ be a proper model for $W$ over $Y$, onto which $D$ extends to a Cartier divisor, and let $\lambda_0$ be the corresponding partial Weil function. Then there is an $M_k$-constant $\gamma$ such that $\lambda_0 \le \lambda_D + \gamma$ on the domain of the partial Weil function. Applying Lemma 5.14 to $X_0$, the extension of $D$, and $\gamma$ then gives a model $X$ and an extension of $D$ to a $X$ such that the corresponding partial Weil function $\lambda'$ satisfies

$$\lambda' \le \max\{0, \lambda_0 - \gamma\} \le \max\{0, \lambda_D\} .$$

This proves the first assertion. The second assertion is immediate from the definitions. □

## §6. Models and heights

Although Theorem 5.9 has the important corollary that Weil functions exist on all proper varieties relative to all Cartier divisors (Theorem 5.11), it has a perhaps more important corollary of allowing a height function on a proper scheme over a function field $k$ to be defined in more geometric terms using a proper model.

To see this, let $Y$ be a model or local model for a global field $k$, let $X$ be a proper scheme over $Y$, and let $D$ be a Cartier divisor on $X$. Let $P$ be an algebraic point on $X$ and let $L = k(P)$. Let $Y'$ be the normalization of $Y$ in $L$; this is a model or local model for $L$. Its closed points correspond bijectively to places of $L$ lying over places of $k$ corresponding to closed points of $Y$. Let $\phi_P \colon Y' \to X$ be the morphism over $Y$ corresponding to the point $P$, let $w$ be a place of $L$ corresponding to a closed point in $Y'$, and let $R_w$ be the valuation ring of $\mathbb{C}_w$. Let $\psi_P \colon \operatorname{Spec} R_w \to X$ be the morphism in $X(R_w)$ corresponding to $P$ at $w$; this is just the composition of $\phi_P$ with the canonical map $\operatorname{Spec} R_w \to Y'$. The image of $\psi_P$ is contained in $\operatorname{Supp} D$ if and only if the image of $\phi_P$ is contained in $\operatorname{Supp} D$. Assume these not to be the case; then the divisor $\phi_P^* D$ is a well-defined divisor on $Y'$, and its pull-back to $\operatorname{Spec} R_w$ equals $\psi_P^* D$. If $\phi_P^* D$ is locally represented on $Y'$ near $w$ by an element $x$, then

$$\lambda_{D,w}(P) = -\log \|\psi_P^* D\|_w = -\log \|x\|_w . \tag{6.1}$$

This latter quantity is independent of the choice of $x$, and (similarly to Definition 5.2) we define:

**Definition 6.2.** Let $D$ be a divisor on a model or local model $Y$ for a local or global field $k$, and let $v$ be a non-archimedean place of $k$ corresponding to a closed point in $Y$, also denoted $v$. Let $x \in k^*$ be a function on $Y$ that locally defines

$D$ near $v$; it is unique up to multiplication by a unit in the local ring $\mathscr{O}_{Y,v}$. Then $\|x\|_v$ is independent of the choice of $x$, and we define

$$\|D\|_v = \|x\|_v \ .$$

Thus, if $\rho\colon \operatorname{Spec} R_v \to Y$ is the canonical map, then $\|D\|_v = \|\rho^* D\|_v$, where the right-hand side uses Definition 5.2. By (6.1), we also have

$$\lambda_{D,w}(P) = -\log \|\phi_P^* D\|_w \ . \tag{6.3}$$

Assume now that $k$ is a function field over a field $F$, or a local field associated to such a function field. If the constant $c$ used in defining the norms on $k$ (**0:**7.7) is equal to $e$, which we now assume, then (6.3) becomes

$$\lambda_{D,w}(P) = \deg_{F,w} \phi_P^* D \tag{6.4}$$

(see Definition **0:**4.9 for the definition of $\deg_{F,w}$). Assume further that $k$ is a function field. Then, summing over all $w \in M_L$ and recalling Definition **2:**6.2, we have

$$h_{\lambda_D,k}(P) = \frac{\deg_F \phi_P^* D}{[L:k]} \ . \tag{6.5}$$

By the product formula, the right-hand side of (6.5) does not change if $D$ is replaced by a linearly equivalent divisor, so if $\mathscr{L} = \mathscr{O}(D)$, then

$$h_{\lambda_D,k}(P) = \frac{\deg_F \phi_P^* \mathscr{L}}{[L:k]} \ , \tag{6.6}$$

for all $P \in X(\bar{k})$ (including $P \in \operatorname{Supp} D$). Thus, in the function field case, given a line sheaf $\mathscr{L}$ on $X$, we may define a function $h_{\mathscr{L},k}\colon X(\bar{k}) \to \mathbb{R}$. The above argument using Weil functions and Definition **2:**6.2 does not prove that $h_{\mathscr{L},k}$ is a height function for $\mathscr{L}\big|_{X_k}$, since $\mathscr{L}$ may not be of the form $\mathscr{O}(D)$ for a Cartier divisor $D$ on $X$. But, in fact, this is the case:

**Theorem 6.7.** *Let $k$ be a function field over a field $F$, and let $Y$ be a complete model for $k$. Assume that $c = e$ in (**0:**7.7). Then the function $h_{\mathscr{L},k}$ defined in (6.6) has the following properties:*

(a). *Let $X$ be a proper scheme over $Y$, and let $\mathscr{L}_1$ and $\mathscr{L}_2$ be line sheaves on $X$. Then $h_{\mathscr{L}_1 \otimes \mathscr{L}_2,k} = h_{\mathscr{L}_1,k} + h_{\mathscr{L}_2,k}$.*

(b). *Let $X_1$ and $X_2$ be proper schemes over $Y$, let $f\colon X_1 \to X_2$ be a morphism over $Y$, and let $\mathscr{L}$ be a line sheaf on $X_2$. Then $h_{f^*\mathscr{L},k} = h_{\mathscr{L},k} \circ f$.*

(c). *If $X = \mathbb{P}^n_Y$ then $h_{\mathscr{O}(1),k} = h_k$.*

(d). *Let $X$ be a proper scheme over $Y$, and let $\mathscr{L}$ be a line sheaf on $X$ which admits a global section $\sigma$ that does not vanish anywhere on $X_k$. Then $h_{\mathscr{L},k} \geq 0$.*

(e). *Let $X$ be a proper scheme over $Y$, and let $\mathscr{L}_1$ and $\mathscr{L}_2$ be line sheaves on $X$ whose restrictions to $X_k$ are isomorphic. Then $h_{\mathscr{L}_1,k} = h_{\mathscr{L}_2,k} + O(1)$.*

(f). *Let $X$ be a proper scheme over $Y$, and let $\mathscr{L}$ be a line sheaf on $X$. Then $h_{\mathscr{L},k}$ is a height function for $\mathscr{L}\big|_{X_k}$.*

*Proof.* Parts (a)–(c) are trivial verifications. Note that they hold exactly, not up to $O(1)$.

For part (d), let $P \in X(\bar{k})$, let $L = k(P)$, let $Y'$ be a maximal model for $L$, and let $\phi_P \colon Y' \to X$ be the morphism corresponding to $P$. Then $\phi_P^* \sigma$ is a nonzero global section of $\phi_P^* \mathscr{L}$; hence

$$\deg_F \phi_P^* \mathscr{L} = \deg_F \phi_P^* \sigma \geq 0 ,$$

and therefore $h_{\mathscr{L},k}(P) \geq 0$ by (6.6).

Next consider part (e). By additivity, it will suffice to show that if $\mathscr{L}$ is a line sheaf on $X$ whose restriction to $X_k$ is trivial, then $h_{\mathscr{L},k} = O(1)$. Let $\sigma$ be the global section of $\mathscr{L}\big|_{X_k}$ corresponding to the section $1 \in \mathscr{O}_{X_k}$ under the given isomorphism; this extends to a section $\sigma$ of $\mathscr{L}$ over an open subset $U$ of $X$ containing the generic fiber.

Let $V = \operatorname{Spec} B$ be an open subset of $X$ on which $\mathscr{L}$ is trivial; we may assume that $V$ lies over an open affine $\operatorname{Spec} A$ of $Y$. Then $\sigma$ corresponds to an element of $B \otimes_A k$; therefore there exists a nonzero element $a \in A$ such that $a\sigma$ lies in $B$. In geometrical language, there is an effective divisor $E_V$ on $Y$ such that $\sigma\big|_{U \cap V}$, tensored with the pull-back to $U \cap V$ of the canonical section of $\mathscr{O}(E_V)$ on $Y$, extends to a section of $\mathscr{L} \otimes \pi^* \mathscr{O}(E_V)$ over all of $U$, where $\pi$ is the morphism $X \to Y$. Let $E$ be the maximum of such $E_V$ as $V$ passes over a finite set of such affine subsets covering $X$. Then $\mathscr{L} \otimes \pi^* \mathscr{O}(E)$ admits a global section which vanishes nowhere on the generic fiber. By parts (a), (d), and (b), and the fact that the variety $\operatorname{Spec} k$ has only one point, we then have

$$h_{\mathscr{L},k}(P) \geq -h_{\pi^* \mathscr{O}(E),k}(P) = -h_{\mathscr{O}(E),k}(\pi(P)) = O(1) .$$

The opposite inequality holds by applying the above argument to $\mathscr{L}^\vee$.

Finally, part (f) is an application of the Height Machine (Theorem **1:**5.18). Indeed, given a proper scheme $X_k$ over $k$ and a line sheaf $\mathscr{L}_k$ on $X_k$, there exists a proper model $X$ for $X_k$ such that $\mathscr{L}_k$ extends to a line sheaf $\mathscr{L}$ on $X$, by Theorem 2.11. This gives a function $X(\bar{k}) \to \mathbb{R}$, which is well-defined modulo $O(1)$ by Corollary 2.3 and parts (b) and (e). By parts (a)–(c), this class of functions satisfies the conditions of the Height Machine, so part (f) follows. □

In the number field case, this does not work. Indeed, if $Y = \operatorname{Spec} R$ is a maximal model for a number field $k$, then line sheaves on $Y$ do not have a degree function. One could, of course, define the degree of a nonzero rational section $\sigma$ of a line sheaf $\mathscr{L}$ on $Y$, or (equivalently) the degree of a divisor on $Y$, using the norms at non-archimedean places of $k$. However, the degree of the nonzero rational section would depend on the choice of $\sigma$ and not just on $\mathscr{L}$, and the degree of the divisor would change if one passed to a linearly equivalent divisor. Both of these facts follow from the fact that there is no product formula for places corresponding to closed points of $Y$: the product formula must also involve the archimedean places, which are not represented in $Y$.

The goal of Arakelov theory will be to find a way to "complete" $Y$ by formally adding the archimedean places.

## References

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