# CHAPTER 1
# HEIGHTS

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13 February 2021

The theory of heights provides a framework for discussing the complexity of a rational point on a scheme (or variety) over $k$; it was introduced by Weil [when?].

As an example, let $x$ be a rational number. Write $x$ as a ratio $p/q$ of two relatively prime integers with $q \neq 0$. Then the (relative, multiplicative) **height** of $x$ is defined as

$$H_{\mathbb{Q}}(x) = \max(|p|, |q|). \tag{0.1}$$

It is immediately clear that this function has the property that for all $C$, the set $\{x \in \mathbb{Q} \mid H_{\mathbb{Q}}(x) \leq C\}$ is finite. This is true also for heights in general (when talking about points on a scheme as opposed to elements of a number field, it is necessary to choose a line sheaf on that scheme before talking about heights; that line sheaf must then be ample for the finiteness result to hold). Heights also satisfy certain functorial properties; these are summed up in the **height machine**, Theorems 5.11 and 5.17.

As a sample application of heights, consider the following theorem of Roth (covered in more detail in Chapter ———).

**Theorem 0.2.** *Let $\alpha \in \overline{\mathbb{Q}}$, let $\epsilon > 0$, and let $c > 0$. Then at most finitely many rational numbers $p/q \in \mathbb{Q}$ (written in lowest terms) satisfy the inequality*

$$\left| \frac{p}{q} - \alpha \right| \leq \frac{c}{|q|^{2+\epsilon}} \ . \tag{0.2.1}$$

We can replace the inequality (0.2.1) with

$$\left| \frac{p}{q} - \alpha \right| \leq \frac{c}{H_{\mathbb{Q}}(p/q)^{2+\epsilon}} \ ,$$

since if $p/q$ is close to $\alpha$, then replacing $|q|$ with $\max\{|p|, |q|\}$ changes the right-hand side by at most a bounded factor.

As a sample application of this theorem, consider the diophantine equation

$$x^3 - 2y^3 = 3\,, \qquad x, y \in \mathbb{Z} \tag{0.3}$$

First, it is clear that for any solution $(x, y)$, $x$ and $y$ must be relatively prime. Rewriting the equation gives

$$\frac{x}{y} - \sqrt[3]{2} = \frac{3}{y(x^2 + xy\sqrt[3]{2} + y^2\sqrt[3]{4})}$$

and therefore

$$\left|\frac{x}{y} - \sqrt[3]{2}\right| \le \frac{c}{\max\{|x|, |y|\}^3}$$

for some $c > 0$. By Roth's theorem, only finitely many rational numbers $x/y$ satisfy this inequality; hence (0.3) has only finitely many solutions.

*Throughout this chapter, unless otherwise specified, $k$ is a number field or a function field, and $M_k$ is its given set of places.*

## §1. Heights on number fields and function fields

We start by defining the height of an element of a number field or function field $k$ and by proving some of its properties.

**Definition 1.1.** For $x \in k$ the **multiplicative height** and **logarithmic height** of $x$ are defined by

$$H_k(x) = \prod_{v \in M_k} \max(1, \|x\|_v) \tag{1.1.1}$$

and

$$h_k(x) = \log H_k(x) = \sum_{v \in M_k} \log \max(1, \|x\|_v),$$

respectively.

**Proposition 1.2.** *If $k = \mathbb{Q}$ then this definition coincides with the height defined by (0.1).*

*Proof.* As above, write $x = p/q$. Since $q \ne 0$, the product formula implies that $\prod_{v \in M_k} \|q\|_v = 1$. Also, since $(p, q) = 1$, $\max(\|p\|_v, \|q\|_v) = 1$ for all non-archimedean $v \in M_{\mathbb{Q}}$ (*i.e.*, for all $v \ne \infty$). These two facts give

$$\begin{aligned}\prod_{v \in M_{\mathbb{Q}}} \max\left(1, \left\|\frac{p}{q}\right\|_v\right) &= \prod_{v \in M_{\mathbb{Q}}} \max(\|p\|_v, \|q\|_v) \\ &= \max(\|p\|_\infty, \|q\|_\infty) \\ &= \max(|p|, |q|).\end{aligned}$$

Therefore the two definitions coincide. □

Formula (0.1) has an analogue for function fields. Let $F$ be a field, let $x \in F(T)$, and write $x = p(T)/q(T)$ for polynomials $p, q \in F[T]$ with $q \neq 0$ and $(p,q) = (1)$. Then $H_{F(T)}(x) = \exp(\max(\deg p, \deg q))$; the proof of this fact is analogous to the proof of Proposition 1.2.

The following lemma describes how the height varies with $k$.

**Lemma 1.3.** *Let $L$ be a finite extension of $k$, let $x_1, \dots, x_n \in k$, and let $v \in M_k$. Then*

$$\prod_{\substack{w \in M_L \\ w|v}} \max(\|x_1\|_w, \dots, \|x_n\|_w) = \max(\|x_1\|_v, \dots, \|x_n\|_v)^{[L:k]}.$$

*Proof.* By (**0:**7.11.2), we have

$$\prod_{\substack{w \in M_L \\ w|v}} \|x\|_w = \|x\|_v^{[L:k]} \tag{1.3.2}$$

for all $x \in k$. But, for each $w|v$ as above, $\|x\|_w$ is a positive power of $\|x\|_v$; therefore $\|x_i\|_w \geq \|x_j\|_w$ if and only if $\|x_i\|_v \geq \|x_j\|_v$. Thus (1.3.2) implies

$$\begin{aligned}\prod_{\substack{w \in M_L \\ w|v}} \max(\|x_1\|_w, \dots, \|x_n\|_w) &= \max\bigl(\|x_1\|_v^{[L:k]}, \dots, \|x_n\|_v^{[L:k]}\bigr) \\ &= \max(\|x_1\|_v, \dots, \|x_n\|_v)^{[L:k]}.\end{aligned}$$

□

**Lemma 1.4.** *Let $L$ be a finite extension of $k$. Then*

$$H_L(x) = H_k(x)^{[L:k]} \qquad \text{for all } x \in k.$$

*Proof.* By Lemma 1.3,

$$H_L(x) = \prod_{v \in M_k} \prod_{\substack{w \in M_L \\ w|v}} \max(1, \|x\|_w) = \prod_{v \in M_k} \max(1, \|x\|_v)^{[L:k]} = H_k(x)^{[L:k]}.$$

□

This leads to an definition of height for elements of the algebraic closure $\bar{k}$ of $k$:

**Definition 1.5.** Let $x \in \bar{k}$. Then the **multiplicative height** and **logarithmic height** of $x$ are defined by

$$H_k(x) = H_{k(x)}(x)^{1/[k(x):k]}$$

and

$$h_k(x) = \log H_k(x) = \frac{1}{[k(x):k]} \log H_{k(x)}(x),$$

respectively.

Lemma 1.4 then implies that for any finite extension field $L$ of $k$ containing $x$,

$$h_k(x) = \frac{1}{[L:k]} h_L(x) = \frac{1}{[L:k]} \sum_{v \in M_k} \log \max(1, \|x\|_v). \tag{1.6}$$

**Remark 1.7.** Often, the subscript $k$ is omitted from the notations $H_k(\cdot)$ or $h_k(\cdot)$; it is then customary to state whether the implicit field is $k(x)$ (often called the **relative multiplicative height** or **relative logarithmic height**) or $\mathbb{Q}$ (often called the **absolute multiplicative height** or **absolute logarithmic height**). For the purposes of this book, the absolute heights are often the more convenient choice, but they have an obvious disadvantage when working with function fields. Therefore, we will often take heights relative to a fixed number field or function field $k$, often the field over which the variety in question is given.

## §2. Northcott's finiteness theorem over number fields

A key aspect of the theory of heights is that the set of elements of a given number field or function field $k$ whose height is below a given bound $C$ is finite if $k$ is a number field, or parameterized by a scheme of finite type over the field of constants, if $k$ is a function field. In fact, more is true: any set of algebraic points of bounded degree over $k$ is has a similar property. This fact was proved by Northcott (Theorem 2.8). We begin by discussing the situation over $\mathbb{Q}$, and then over arbitrary number fields. The function field case is proved by a different method; that proof is given in Section 6.

The proof of Northcott's finiteness theorem relies on an older type of height function, which we now introduce.

**Definition 2.1.** Let $f(T) = a_n T^n + \cdots + a_0$ be a nonconstant polynomial in $k[T]$. For all places $v$ of $k$, let

$$\|f\|_v = \max(\|a_0\|_v, \ldots, \|a_n\|_v).$$

This definition does not depend on $n$. Also let

$$H_k(f) = \prod_{v \in M_k} \|f\|_v. \tag{2.1.1}$$

**Definition 2.2.** Let $x \in \bar{k}$ and let $f(T) = \mathrm{Irr}_{x,k}(T)$. Then we define

$$H_{\mathrm{ancient},k}(x) = H_k(f).$$

Comparing this height with the height introduced in Section 1 requires some lemmas on the height $H_k(f)$.

**Lemma 2.3.** *Let* $f(T) = a_n T^n + \cdots + a_0$ *be a nonconstant polynomial in* $k[T]$*, and let* $x \in k^*$*. Then* $H_k(xf) = H_k(f)$*.*

*Proof.* This follows directly from the product formula:

$$H_k(xf) = \prod_{v \in M_k} \max(\|xa_0\|_v, \ldots, \|xa_n\|_v) = \prod_{v \in M_k} \|x\| \max(\|a_0\|_v, \ldots, \|a_n\|_v) = H_k(f).$$

□

**Lemma 2.4.** *Let* $L$ *be a finite extension of* $k$*. Then:*

(a). *If* $f(T) \in k[T]$ *is a nonconstant polynomial, then*

$$H_L(f) = H_k(f)^{[L:k]}.$$

(b). *If* $x \in L$ *and* $\sigma$ *is an automorphism of* $L$ *over* $k$*, then*

$$H_L(x) = H_L(\sigma(x)).$$

*Proof.* Part (a) follows from Lemma 1.3: write $f(T) = a_n T^n + \cdots + a_0$; then

$$\begin{aligned} H_L(f) &= \prod_{v \in M_k} \prod_{\substack{w \in M_L \\ w|v}} \max(\|a_0\|_w, \ldots, \|a_n\|_w) \\ &= \prod_{v \in M_k} \max(\|a_0\|_v, \ldots, \|a_n\|_v)^{[L:k]} \\ &= H_k(f)^{[L:k]}. \end{aligned}$$

Part (b) follows from the fact that $\sigma$ permutes the set $\{w \in M_L \mid w \mid v\}$, and that $\|x\|_w = \|\sigma(x)\|_{\sigma(w)}$. Thus applying $\sigma$ to $x$ just permutes the factors in the expression (1.1.1). □

**Lemma 2.5.** *Let* $v$ *be a place of* $k$ *and let*

$$f(T) = \prod_{i=1}^{n} (T - x_i) \tag{2.5.1}$$

*be a nonconstant polynomial in* $k[T]$*. Then*

$$2^{-nN_v} \prod_{i=1}^{n} \max(1, \|x_i\|_v) \le \|f\|_v \le 2^{nN_v} \prod_{i=1}^{n} \max(1, \|x_i\|_v), \tag{2.5.2}$$

*where* $N_v$ *is as defined in* §**0:***7.*

*Proof.* First consider the non-archimedean case. Gauss' lemma ([**L**], Ch. IV, Theorem 2.1) asserts that if $f, g \in k[T]$ then $\|fg\|_v = \|f\|_v \|g\|_v$. Applying this to the factorization (2.5.1) then gives

$$\|f\|_v = \prod_{i=1}^{n} \|T - x_i\|_v = \prod_{i=1}^{n} \max(1, \|x_i\|_v).$$

This is equivalent to (2.5.2).

To prove the archimedean case, it suffices to prove the following statement. For a polynomial $f(T) = a_n T^n + \cdots + a_0$ in $\mathbb{C}[T]$, let $|f| = \max(|a_0|, \ldots, |a_n|)$. Let $x_1, \ldots, x_n \in \mathbb{C}$ and let $f(T)$ be the polynomial defined by (2.5.1). Then

$$2^{-n} \prod_{i=1}^{n} \max(1, |x_i|) \le |f| \le 2^n \prod_{i=1}^{n} \max(1, |x_i|).$$

The second half is easy, since the coefficient $a_m$ of $T^m$ in $f$ is (up to sign) the sum of $\binom{n}{m} \le 2^n$ terms, each of which is a product of distinct elements of the tuple $(x_1, \ldots, x_n)$.

To prove the first half, let $d$ be the number of indices $i$ such that $|x_i| \ge 2$. We will prove the inequality by induction on $d$. If $d = 0$ then the inequality is easy, since $|f| \ge 1$. Otherwise, without loss of generality we may assume that $|x_n| \ge 2$. Write

$$g(T) = \prod_{i=1}^{n-1} (T - x_i) = b_{n-1} T^{n-1} + \cdots + b_0.$$

Then $f(T) = (T - x_n) g(T)$, so

$$f(T) = b_{n-1} T^n + (b_{n-2} - x_n b_{n-1}) T^{n-2} + \cdots + (b_0 - x_n b_1) T - x_n b_0.$$

Choose some index $j$ such that $|g| = |b_j|$. Then, letting $b_{-1} = 0$ if $j = 0$, we have

$$\begin{aligned}
|f| &\ge |x_n b_j - b_{j-1}| \\
&\ge |x_n||b_j| - |b_{j-1}| \\
&\ge |x_n||g| - |g| \\
&\ge \frac{|x_n|}{2} |g| \\
&\ge \frac{\max(1, |x_n|)}{2} \cdot 2^{-(n-1)} \prod_{i=1}^{n-1} \max(1, |x_i|) \\
&\ge 2^{-n} \prod_{i=1}^{n} \max(1, |x_i|).
\end{aligned}$$

Translating back in terms of $\|\cdot\|_v$ then gives (2.5.2). □

We are now ready to compare the ancient height with the one defined in Section 1. It is then an easy step to prove Northcott's finiteness theorem.

**Proposition 2.6.** *Let* $x \in \bar{k}$.

(a). *If* $k$ *is a number field, then*

$$2^{-[k(x):\mathbb{Q}]} H_{k(x)}(x) \le H_{\mathrm{ancient},k}(x) \le 2^{[k(x):\mathbb{Q}]} H_{k(x)}(x).$$

(b). *If* $k$ *is a function field, then*

$$H_{\mathrm{ancient},k}(x) = H_{k(x)}(x).$$

*Proof.* Let $L$ be a finite normal extension field of $k$ that contains $x$. Let $x_1, \dots, x_n$ be the conjugates of $x$ over $k$ in $L$, each repeated $[k(x):k]_{\mathrm{i}}$ times, so that $n = [k(x):k]$ and

$$f(T) := \prod_{i=1}^{n} (T - x_i)$$

is the irreducible polynomial of $x$ over $k$. Taking the product of (2.5.2) over all $v \in M_L$ gives

$$\left( \prod_{v \in M_L} 2^{N_v} \right)^{-n} \prod_{i=1}^{n} H_L(x_i) \le H_L(f) \le \left( \prod_{v \in M_L} 2^{N_v} \right)^{n} \prod_{i=1}^{n} H_L(x_i).$$

By Lemma 2.4b, $H_L(x_i) = H_L(x)$ for all $i$; hence

$$\left( \prod_{v \in M_L} 2^{N_v} \right)^{-n} H_L(x)^n \le H_L(f) \le \left( \prod_{v \in M_L} 2^{N_v} \right)^{n} H_L(x)^n.$$

The proposition then follows from the facts that

$$\begin{aligned} H_L(f) &= H_{\mathrm{ancient},k}(x)^{[L:k]}, \\ H_L(x)^n &= H_{k(x)}(x)^{[L:k]}, \end{aligned}$$

and

$$\sum_{v \in M_L} N_v = \begin{cases} [L:\mathbb{Q}] & \text{if } k \text{ is a number field;} \\ 0 & \text{if } k \text{ is a function field.} \end{cases} \tag{2.6.1}$$

□

The above proof exhibits a pattern that frequently occurs: the non-archimedean case is relatively easy, and holds exactly, whereas the archimedean case is harder and holds only up to a constant.

**Remark 2.7.** When stated with logarithmic heights (which are defined in the obvious way from the corresponding multiplicative heights), and with the use of the convention

**0:**7.8 that $[k:\mathbb{Q}]=0$ when $k$ is a function field, both parts of Proposition 2.6 can be combined into one statement:

$$\left|h_{k(x)}(x)-h_{\mathrm{ancient},k}(x)\right|\le[k(x):\mathbb{Q}]\log 2.$$

Likewise, (2.6.1) can be shortened to

$$\sum_{v\in M_L}N_v=[L:\mathbb{Q}].$$

In the sequel, theorems will be stated in this way.

We are now ready to prove the main theorem of the section.

**Theorem 2.8** (Northcott). *Let $k$ be a number field, let $d\in\mathbb{Z}_{>0}$, and let $C\in\mathbb{R}$. Then the set*

$$\{x\in\bar{k}\mid[k(x):k]\le d \text{ and } H_{k(x)}(x)\le C\}$$

*is finite.*

*Proof.* Let $\Sigma$ denote the above set. If $C<1$ then $\Sigma=\emptyset$, so we may assume that $C\ge 1$. By replacing $k$ with $\mathbb{Q}$, multiplying $d$ by $[k:\mathbb{Q}]$, and replacing $C$ with $C^{1/[k:\mathbb{Q}]}$, we may assume that $k=\mathbb{Q}$.

By Proposition 2.6, we have

$$\Sigma\subseteq\{x\in\overline{\mathbb{Q}}\mid[\mathbb{Q}(x):\mathbb{Q}]\le d \text{ and } H_{\mathrm{ancient},\mathbb{Q}}(x)\le 2^{[\mathbb{Q}(x):\mathbb{Q}]}C\}.$$

But this latter set is finite, since $\mathrm{Irr}_{x,\mathbb{Q}}(T)$ has degree at most $d$ and, after cancelling denominators, its coefficients are integers bounded in absolute value by $2^{[\mathbb{Q}(x):\mathbb{Q}]}C$. □

**Remark 2.9.** The restriction on the degree cannot be dropped: the set of roots of unity in $\overline{\mathbb{Q}}$ is infinite, but all roots of unity have (multiplicative) height $1$. (The converse also holds: if $x\in\overline{\mathbb{Q}}$ has $H_{\mathbb{Q}(x)}(x)=1$, then $x$ is a root of unity.)

**Remark 2.10.** Northcott's theorem only used an upper bound on $H_{\mathrm{ancient},k}(x)$ in terms of $H_{k(x)}(x)$; this in turn relies only on the second (easier) half of (2.5.2). But the ancient height occasionally comes up, and it is useful to know both bounds.

## §3. Heights on projective space

The ultimate goal of the theory of heights is to define the heights of closed points on a scheme of finite type over $k$; the first step in doing so is to define the height of a closed point on $\mathbb{P}^n_k$. In later sections this will be extended to more general schemes (using Chow's lemma if the scheme is not projective).

**Definition 3.1.** Let $n \in \mathbb{N}$ and let $P \in \mathbb{P}^n_k(k)$. Let $[x_0 : \cdots : x_n]$ be homogeneous coordinates for $P$, with $x_i \in k$ for all $i$. Then the **multiplicative height** and **logarithmic height** of $P$ are

$$H_k(P) = \prod_{v \in M_k} \max(\|x_0\|_v, \ldots, \|x_n\|_v).$$

and

$$h_k(P) = \log H_k(P) = \sum_{v \in M_k} \log \max(\|x_0\|_v, \ldots, \|x_n\|_v),$$

respectively.

This definition does not depend on the choice of homogeneous coordinates. Indeed, let $[y_0 : \cdots : y_n]$ be any other set of homogeneous coordinates for $P$ with $y_i \in k$ for all $i$. Then there exists $w \in k^*$ such that $y_i = wx_i$ for all $i$; therefore

$$\begin{aligned}\prod_{v \in M_k} \max(\|y_0\|_v, \ldots, \|y_n\|_v) &= \prod_{v \in M_k} \|w\|_v \cdot \prod_{v \in M_k} \max(\|x_0\|_v, \ldots, \|x_n\|_v) \\ &= \prod_{v \in M_k} \max(\|x_0\|_v, \ldots, \|x_n\|_v)\end{aligned}$$

by the product formula applied to $w$.

**Lemma 3.2.** *Let $k$ be a number field or function field.*

(a). *For all $x \in \bar{k}$, the height of $x$ and the heights of the corresponding point $[1 : x] \in \mathbb{P}^1_k$ are equal: $H_L(x) = H_L([1 : x])$ for all global fields $L$ containing $x$ and extending $k$.*

(b). *If $L$ is a finite extension of $k$ and if $n \in \mathbb{N}$ then*

$$H_L(P) = H_k(P)^{[L:k]}$$

*for all $P \in \mathbb{P}^n_k(k)$.*

*Proof.* Part (a) follows directly by comparing Definitions 1.1 and 3.1. Part (b) follows from Lemma 1.3. □

As before, we may then extend the definition to give heights of algebraic points.

**Definition 3.3.** Let $n \in \mathbb{N}$ and let $P \in \mathbb{P}^n(\bar{k})$. Then the **multiplicative height** of $P$ is defined by

$$H_k(P) = H_{k(P)}(P)^{1/[k(P):k]},$$

and the **logarithmic height** is defined similarly.

Again, we could use any finite extension $L \supseteq k(P)$ in place of $k(P)$ in the above definition.

Northcott's finiteness theorem also holds in the context of points in projective space. Its proof follows rather easily from the corresponding result for heights of elements of $\bar{k}$.

**Theorem 3.4** (Northcott). *Let $k$ be a number field, let $n \in \mathbb{N}$, let $d \in \mathbb{Z}_{>0}$, and let $C \in \mathbb{R}$. Then the set*

$$\{P \in \mathbb{P}^n(\bar{k}) \mid [k(P) : k] \le d \text{ and } H_{k(P)}(P) \le C\}$$

*is finite.*

*Proof.* Let $\Sigma$ denote the above set. Without loss of generality, it will suffice to prove the assertions for the set

$$\Sigma' := \Sigma \cap \{P \in \mathbb{P}^n(\bar{k}) \mid x_0(P) \ne 0\}.$$

Then the result follows from the fact that

$$h((x_i/x_0)(P)) \le h(P)$$

for all $i = 1, \dots, n$ and all $P \in \mathbb{P}^n(\bar{k})$, via the earlier Northcott theorem 2.8. □

## §4. Heights and linear maps

As is common in algebraic geometry, it is often better to work with heights on an abstract variety than with heights on a variety regarded as a subset of projective space. In doing this, it will become clear that some additional data are needed: at the very least one must fix a line sheaf. Even after doing so, however, the logarithmic height is only defined up to addition of a bounded function. The easiest way to see this is that the automorphism group of $\mathbb{P}^1_k$ is large enough to take any given rational point to any other rational point, so the height may vary by an arbitrarily large amount, depending on the automorphism.

The purpose of this section is to show that things are not any worse: the logarithmic height on $\mathbb{P}^n$ changes by at most a bounded function when applying an automorphism. This fact will be used in Section 5 to attach a height to a suitable set of global sections of a line sheaf $\mathscr{L}$.

From now on we will work primarily with logarithmic heights. This is easier from a notational point of view; in addition, logarithmic heights can be expressed as intersection numbers.

We begin by formulating a definition which will be used a few times in this chapter and will play a much larger role in the chapter on Weil functions.

**Definition 4.1.** Let $k$ be either a number field, a function field, or the completion of a number field or function field at one of its given places. Then an $M_k$**-constant** is a function $\gamma\colon M_k \to \mathbb{R}$, written $v \mapsto \gamma_v$, such that $\gamma_v = 0$ for almost all $v$. We extend $\gamma$ to a map on $\overline{M}_k$ by letting

$$\gamma_w = [L_w : k_v]\gamma_v \tag{4.1.1}$$

if $w \in M_L$ lies over $v \in M_k$. The set of $M_k$-constants forms a group under componentwise addition. We say say that $\gamma \geq 0$ if $\gamma_v \geq 0$ for all $v$, and $\gamma \geq \gamma'$ if $\gamma - \gamma' \geq 0$. Finally, we let

$$|\gamma| = \sum_{v \in M_k} \gamma_v.$$

Note that if $\gamma$ is an $M_k$-constant and if $L$ is a finite extension of $k$, then

$$|\gamma| = \frac{1}{[L:k]} \sum_{w \in M_L} \gamma_w. \tag{4.2}$$

Of course, if $k$ the completion of a number field or function field at one of its given places, then an $M_k$-constant is just a constant, but it is convenient to have the same terminology. Such constants do not occur in this chapter, but will occur later on.

**Example 4.3.** Some examples of $M_k$-constants are:

(a). Recall from **0:**7.2 that $N_v$ is defined to be $0$ if $v$ is non-archimedean, $1$ if $v$ is real, and $2$ if $v$ is complex. This defines an $M_k$-constant $N$.

(b). If $a_1, \ldots, a_n \in k$ and not all are zero, then $\log \max\{\|a_1\|_v, \ldots, \|a_n\|_v\}$ is an $M_k$-constant.

Note that in both of these examples, the defining formulas are compatible with (4.1.1).

**Lemma 4.4.** *Let $n \in \mathbb{N}$ and let $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$. Then*

$$h_k(\phi(P)) = h_k(P) + O(1) \qquad \text{for all } P \in \mathbb{P}^n(\bar{k})$$

*where the implicit constant in the $O(1)$ term depends only on $k$ and $\phi$*

*Proof.* By ([**H 2**], II 7.1.1), $\mathrm{Aut}(\mathbb{P}^n_k) \cong \mathrm{PGL}_{n+1}(k)$. Therefore there exists a nonsingular $(n+1)\times(n+1)$ matrix $M$ with entries in $k$ such that, $\phi$ takes a point $P \in \mathbb{P}^n_k$ with homogeneous coordinates $[x_0 : \cdots : x_n]$ to a point with homogeneous coordinates $[y_0 : \cdots : y_n]$, where

$$\begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = M \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}. \tag{4.4.1}$$

The main part of the proof consists of showing that there are $M_k$-constants $\gamma$ and $\gamma'$ such that all $P \in \mathbb{P}^n(\bar{k})$, all $L \supseteq k(P)$, and all $w \in M_L$ have the following property. Let $[x_0 : \cdots : x_n]$ be homogeneous coordinates for $P$ with $x_0, \ldots, x_n \in L$ and let $y_0, \ldots, y_n$ be defined by (4.4.1). Then

(4.4.2)

$$e^{-\gamma'_w} \max(\|x_0\|_w, \ldots, \|x_n\|_w) \leq \max(\|y_0\|_w, \ldots, \|y_n\|_w) \leq e^{\gamma_w} \max(\|x_0\|_w, \ldots, \|x_n\|_w).$$

Indeed, let $v$ be a place of $k$. For $(n+1)\times(n+1)$ matrices $A=(a_{ij})_{0\le i,j\le n}$ with entries in $k$, let

$$\|A\|_v = \max_{0\le i,j\le n} \|a_{ij}\|_v.$$

Let $L$, $w$, $P$, and $x_0,\dots,x_n$ be as above. Then, by (4.4.1) and **0:**7.1,

$$\begin{aligned}\max_{0\le i\le n}\|y_i\|_w &= \max_{0\le i\le n}\left\|\sum_{j=0}^n m_{ij}x_j\right\|_w \\ &\le (n+1)^{N_v}\max_{0\le i\le n}\max_{0\le j\le n}\|m_{ij}x_j\|_w \\ &\le (n+1)^{N_v}\|M\|_w\max_{0\le j\le n}\|x_j\|_w\ ,\end{aligned}$$

where $m_{ij}$ denote the entries of $M$. This proves the second half of (4.4.2), with $\gamma_v=\log\|M\|_v$. The first half is similar, with $\gamma'_v=\log\|M^{-1}\|_v$. From the definitions it is clear that $(\gamma_v)_{v\in M_k}$ and $(\gamma'_v)_{v\in M_k}$ are $M_k$-constants.

Thus (4.4.2) holds. Taking logarithms of all sides and summing over all $w\in M_L$ gives the inequality

$$-|\gamma'|\le h_k(\phi(P))-h_k(P)\le|\gamma|$$

for all $P$. □

**Lemma 4.5.** *Let $n\in\mathbb{Z}_{>0}$. Let $V$ and $W$ be linear subspaces of $\mathbb{P}^n_k$ (hence defined by linear equations with coefficients in $k$) such that $V\cap W=\emptyset$ and such that $\dim V+\dim W=n-1$. Let $\phi\colon\mathbb{P}^n_k\backslash V\to W$ be the corresponding linear projection. Let $X$ be a closed subscheme of $\mathbb{P}^n_k$ such that $X\cap V=\emptyset$. Then*

$$h_k(\phi(P))=h_k(P)+O(1) \tag{4.5.1}$$

*for all $P\in X(\bar k)$.*

*Proof.* By Lemma 4.4 we may assume that $V$ is defined by $x_0=\cdots=x_m=0$ and $W$ by $x_{m+1}=\cdots=x_n=0$. Then $\phi$ is the morphism

$$[x_0:\cdots:x_n]\mapsto[x_0:\cdots:x_m:0:\cdots:0]. \tag{4.5.2}$$

Then half of (4.5.1) is easy. Indeed, let $L$ be a finite extension of $k$ and let $P\in X(L)$. Choose homogeneous coordinates $[x_0:\cdots:x_n]$ for $P$ with $x_i\in L$ for all $i$. Then, obviously,

$$\max(\|x_0\|_w,\dots,\|x_m\|_w)\le\max(\|x_0\|_w,\dots,\|x_n\|_w)$$

for all $w\in M_L$; hence $H_L(\phi(P))\le H_L(P)$ and therefore $h_k(\phi(P))\le h_k(P)$.

The other half of (4.5.1) is a little harder. We first prove the special case $m=n-1$. In that case $\phi$ is the linear projection from the point $Q=[0:\cdots:0:1]$ to the hyperplane $W=\{x_n=0\}$.

We first claim that there is an $M_k$-constant $\gamma$ such that

$$\|x_n\|_w \le e^{\gamma_w} \max\{\|x_0\|_w, \dots, \|x_{n-1}\|_w\} \tag{4.5.3}$$

for all fields $L$ finite over $k$, all $w \in M_L$, and all points $[x_0 : \cdots : x_n] \in X(L)$.

First consider archimedean $v \in M_k$. This corresponds (non-canonically) to an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$. Thinking of $\operatorname{Spec}\mathbb{C}$ as a scheme over $k$ via $\sigma$, we have that $X(\mathbb{C})$ is a subset of $\mathbb{P}^n(\mathbb{C})$, closed and hence compact in the classical topology. Therefore the continuous function

$$\frac{|z_n|}{\max\{|z_0|, \dots, |z_{n-1}|\}}\ , \qquad [z_0 : \cdots : z_n] \in X(\mathbb{C})$$

has a maximum value $B$, and we let $\gamma_v = N_v \log B$. Then also $\gamma_w = N_w \log B$ for all $w$ as in the claim, and (4.5.3) holds for all points in $X(L)$ and all $w \in M_L^\infty$.

Next consider non-archimedean places $v \in M_k$. Note that, although $\mathbb{Q}_p$ and $F((t))$ with $F$ finite are locally compact, their algebraic closures are not, so the above compactness argument does not extend to this case.

Instead, since $Q \notin X$ there is a homogeneous polynomial

$$f(T_0, \dots, T_n) \in k[T_0, \dots, T_n]$$

that vanishes at all points of $X$ but not at $Q$. For multiindices $\mathbf{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$, write $T^{\mathbf{i}} = T_0^{i_0} \cdots T_n^{i_n}$ and $|\mathbf{i}| = i_0 + \cdots + i_n$. Then $f$ can be written

$$f(T) = \sum_{|\mathbf{i}|=d} a_{\mathbf{i}} T^{\mathbf{i}}\ ,$$

where $d = \deg f$. By assumption $f(0, \dots, 0, 1) \ne 0$, so $a_{(0,\dots,0,d)} \ne 0$. After multiplying $f$ by a nonzero constant, we may assume that $a_{(0,\dots,0,d)} = 1$.

Let $P \in X(\bar{k})$, let $L$ be a finite extension of $k$ such that $P \in X(L)$, let $w \in M_L$ be a place lying over $v$, and let $[x_0 : \cdots : x_n]$ be homogeneous coordinates for $P$. Since (4.5.3) is trivial when $x_n = 0$, we may assume (by scaling) that $x_n = 1$; then $x_i \in L$ for all $i$.

Since $f(P) = 0$, we then have

$$1 = a_{(0,\dots,0,d)} x_n^d = -\sum_{\substack{|\mathbf{i}|=d \\ \mathbf{i} \ne (0,\dots,0,d)}} a_{\mathbf{i}} x_0^{i_0} \cdots x_n^{i_n}\ .$$

Since $v$ and hence $w$ is non-archimedean, some term on the right-hand side must have norm (at $w$) at least $1$; in other words, there exists an $\mathbf{i} \in \mathbb{N}^{n+1}$ such that $|\mathbf{i}| = d$, $\mathbf{i} \ne (0, \dots, 0, d)$, and $\|a_{\mathbf{i}} x_0^{i_0} \cdots x_n^{i_n}\|_w \ge 1$. Then

$$\|a_{\mathbf{i}}\|_w \max\{\|x_0\|_w, \dots, \|x_{n-1}\|_w\}^{i_0 + \cdots + i_{n-1}} \ge \|a_{\mathbf{i}} x_0^{i_0} \cdots x_n^{i_n}\|_w \ge 1\ ,$$

which implies

$$\log\max\{\|x_0\|_w,\dots,\|x_{n-1}\|_w\} + \max_{\substack{|\mathbf{i}|=d\\ \mathbf{i}\neq(0,\dots,0,d)\\ a_{\mathbf{i}}\neq 0}} \frac{\log\|a_{\mathbf{i}}\|_w}{i_0+\cdots+i_{n-1}} \geq 0\,.$$

Since $x_n = 1$ and $a_{(0,\dots,0,d)} = 1$, (4.5.3) holds for places $w \mid v$, where

$$\gamma_v = \max_{\substack{|\mathbf{i}|=d\\ \mathbf{i}\neq(0,\dots,0,d)}} \frac{\log\|a_{\mathbf{i}}/a_{(0,\dots,0,d)}\|_v}{i_0+\cdots+i_{n-1}}\,. \tag{4.5.4}$$

Since $\gamma_v = 0$ for all but finitely many $v \in M_k$, $\gamma$ is an $M_k$-constant satisfying (4.5.3), so the claim is proved.

Let $\gamma' = \max(0,\gamma)$; this is also an $M_k$-constant. By (4.5.3),

$$\max(\|x_0\|_w,\dots,\|x_n\|_w) \leq e^{\gamma'_w}\max(\|x_0\|_w,\dots,\|x_{n-1}\|_w)$$

for all $L$, $w$, and $P = [x_0 : \cdots : x_n]$ as above. Taking the logarithm and summing over all $w \in M_L$ then gives

$$h_L(P) \leq h_L(\phi(P)) + [L:k]\,|\gamma'|$$

for all $P$ and $L$. Dividing by $[L:k]$ then gives the other half of (4.5.1), in the special case $m = n-1$.

The general case of the lemma is proved by induction on $n-m$. The base case $n-m=1$ was proved just above.

For the inductive step, assume that $n-m>1$, and factor $\phi\colon \mathbb{P}^n_k \setminus V \to W$ as the composition $\phi = \phi' \circ \big(\phi_0\big|_{\mathbb{P}^n_k\setminus V}\big)$, as follows. Let $W_0 \subseteq \mathbb{P}^n_k$ be the hyperplane $\{x_n = 0\}$, let $Q = [0:\cdots:0:1]$ be as above, and let $\phi_0\colon \mathbb{P}^n_k\setminus\{Q\} \to W_0$ be the map $[x_0:\cdots:x_n] \mapsto [x_0:\cdots:x_{n-1}:0]$ (as in the special case $n-m=1$). Let $\phi'\colon W_0\setminus V' \to W$ be the map $[x_0:\cdots:x_{n-1}:0] \mapsto [x_0:\cdots:x_m:0:\cdots:0]$, where $V' = V\cap W_0$; this is the linear subspace $x_0=\cdots=x_m=x_n=0$.

$$\begin{array}{ccccc}
\mathbb{P}^n_k\setminus\{Q\} & \xrightarrow{\phi_0} & W_0 \cong \mathbb{P}^{n-1}_k & & \\
\cup & & \cup & & \\
X \subseteq \mathbb{P}^n_k\setminus V & \xrightarrow{\phi_0\big|_{\mathbb{P}^n_k\setminus V}} & W_0\setminus V' & \xrightarrow{\phi'} & W
\end{array}$$

Note that $V' = \phi_0(V\setminus\{Q\})$, so that $\phi_0^{-1}(V') = \{x_0=\cdots=x_m=0\}\setminus\{Q\} = V\setminus\{Q\}$. Since $X\cap V = \emptyset$, $X\cap\phi_0^{-1}(V') = \emptyset$; hence $\phi_0(X)\cap V' = \emptyset$. Also $\phi_0(X)$ is a closed subset of $W_0$, since the image of a proper scheme is proper (([**H 1**], II, Ex. 4.4)). Therefore Lemma 4.5 can be applied (by the inductive hypothesis) to $\phi'\colon W_0\setminus V' \to W$

(via the isomorphism $W_0 \cong \mathbb{P}_k^{n-1}$). Combining this with the special case $n-m=1$ applied to $\phi_0\big|_{\mathbb{P}_k^n \setminus V} : \mathbb{P}_k^n \setminus V \to W_0$ gives

$$h_k(\phi(P)) = h_k(\phi'(\phi_0(P))) = h_k(\phi_0(P)) + O(1) = h_k(P) + O(1) \ ,$$

giving (4.5.1). □

**Remark 4.6.** In the above proof it is not possible to use a compactness argument at the non-archimedean places. First of all, if $v$ is non-archimedean, then $\mathbb{P}^n(\bar{k}_v)$ is not compact (and in fact, $\mathbb{P}^n(k_v)$ itself fails to be compact if $k_v$ has infinite residue field). In addition, the bounds $\gamma_v$ must vanish for almost all $v$.

## §5. Heights on schemes

We are now ready to define the height on a projective scheme (or variety) over $k$. The results of Section 4 will enable us to show that such heights are well-defined modulo $O(1)$. But first, we recall some facts concerning global sections of a line sheaf.

*Throughout this section, $X$ is a scheme, separated and of finite type over $k$.*

**Definition 5.1.** Let $\mathscr{L}$ be a line sheaf on $X$.

(a). We say that $\mathscr{L}$ is **globally generated** if it is generated by its global sections.
(b). Let $D$ be a Cartier divisor on $X$. We say that $D$ is **globally generated** if $\mathscr{O}(D)$ is globally generated.
(c). Let $U$ be an open subset of $X$. We say that $\mathscr{L}$ is **globally generated over** $U$ if $\mathscr{L}\big|_U$ is generated by the restrictions of elements of $\Gamma(X, \mathscr{L})$.

Thus $\mathscr{L}$ is globally generated if and only if it is globally generated over $X$.

(If $X$ is nonsingular, $\mathscr{L}$ is globally generated if and only if the associated complete linear system is base-point free—see ([**H 2**], II Lemma 7.8).)

**Remark.** If $\mathscr{L}$ is globally generated over an open set $U$, then there exists a finite set $s_0, \dots, s_n$ of global sections (in $\Gamma(X, \mathscr{L})$) whose restrictions to $U$ generate $\mathscr{L}\big|_U$. This follows from quasi-compactness of $U$.

**Lemma 5.2.** *Let $\mathscr{L}$ be a globally generated line sheaf on $X$ and let $\mathbf{s} = (s_0, \dots, s_n)$ be a system of global sections which generates $\mathscr{L}$. Then there exists a unique morphism $\phi_{\mathbf{s}} : X \to \mathbb{P}_k^n$ such that $\mathscr{L} \cong \phi_{\mathbf{s}}^* \mathscr{O}(1)$, and such that $s_i = \phi_{\mathbf{s}}^* x_i$ under this isomorphism, where $x_0, \dots, x_n$ are the global sections of $\mathscr{O}(1)$ corresponding to the standard homogeneous coordinates on $\mathbb{P}_k^n$.*

*Proof.* See ([**H 2**], II Theorem 7.1b). □

**Definition 5.3.** Let $X$, $\mathscr{L}$, $\mathbf{s}$, and $\phi_{\mathbf{s}}$ be as in Lemma 5.2. Then we define

$$h_{\mathbf{s},k}(P) = h_k(\phi_{\mathbf{s}}(P))$$

for all $P \in X(\bar{k})$, where $h_k$ on the right-hand side is the logarithmic height on $\mathbb{P}^n_k$ defined in Section 3.

This definition will form the basis for the definition of the height on a scheme, relative to a line sheaf. Before making this definition, some lemmas will be needed to show that the height is well defined.

**Lemma 5.4.** *Let $X$ be a proper scheme over $k$ and let $\mathscr{L}$ be a globally generated line sheaf on $X$. Let $\mathbf{s} = (s_0, \dots, s_n)$ and $\mathbf{t} = (t_0, \dots, t_m)$ be two systems of global sections, each of which generates $\mathscr{L}$. Then*

$$h_{\mathbf{s},k} = h_{\mathbf{t},k} + O(1).$$

*Proof.* We first consider the case in which $\mathbf{s}$ is a subset of $\mathbf{t}$. Permuting the elements of $\mathbf{t}$ does not change $h_{\mathbf{t},k}$, so we may assume that $s_i = t_i$ for all $i = 0, \dots, n$ (and therefore that $n \le m$). Let $\phi_{\mathbf{s}} \colon X \to \mathbb{P}^n_k$ and $\phi_{\mathbf{t}} \colon X \to \mathbb{P}^m_k$ be as in Lemma 5.2, and let $\psi \colon \mathbb{P}^m_k \setminus \{x_0 = \cdots = x_n = 0\} \to \mathbb{P}^n_k$ be the linear projection defined in homogeneous coordinates by

$$[x_0 : \cdots : x_m] \mapsto [x_0 : \cdots : x_n].$$

Then $\phi_{\mathbf{s}} = \psi \circ \phi_{\mathbf{t}}$, and we have

$$h_{\mathbf{s},k}(P) = h_k(\phi_{\mathbf{s}}(P)) = h_k(\psi(\phi_{\mathbf{t}}(P))) = h_k(\phi_{\mathbf{t}}(P)) + O(1) = h_{\mathbf{t},k}(P) + O(1)$$

for all $P \in X(\bar{k})$, by Lemma 4.5.

The general case follows by passing from $\mathbf{s}$ to $\mathbf{t}$ via the union $\mathbf{s} \cup \mathbf{t}$. □

**Lemma 5.5.** *Let $\mathscr{L}$ and $\mathscr{M}$ be two globally generated line sheaves on $X$. Let $\mathbf{s} = (s_0, \dots, s_n)$ and $\mathbf{t} = (t_0, \dots, t_m)$ be systems of global sections which generates $\mathscr{L}$ and $\mathscr{M}$, respectively. Then*

$$\mathbf{s} \otimes \mathbf{t} := (s_i \otimes t_j)_{0 \le i \le n,\ 0 \le j \le m}$$

*is a system of global sections which generates $\mathscr{L} \otimes \mathscr{M}$, and*

$$h_{\mathbf{s}\otimes\mathbf{t},k} = h_{\mathbf{s},k} + h_{\mathbf{t},k}. \tag{5.5.1}$$

*Proof.* The first assertion is obvious. Indeed, given $P \in X$, pick $i$ and $j$ such that $s_i$ and $t_j$ generate $\mathscr{L}$ and $\mathscr{M}$, respectively, in a neighborhood of $P$. Then $s_i \otimes t_j$ generates $\mathscr{L} \otimes \mathscr{M}$ in a neighborhood of $P$.

Let $\phi_{\mathbf{s}} \colon X \to \mathbb{P}^n_k$, $\phi_{\mathbf{t}} \colon X \to \mathbb{P}^m_k$, and $\phi_{\mathbf{s}\otimes\mathbf{t}} \colon X \to \mathbb{P}^{nm+n+m}_k$ be the morphisms corresponding to $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{s} \otimes \mathbf{t}$, respectively, by Lemma 5.2. Pick $P \in X(\bar{k})$ and let $L = k(P)$. Let $[x_0 : \cdots : x_n]$ and $[y_0 : \cdots : y_m]$ be homogeneous coordinates for $\phi_{\mathbf{s}}(P)$ and $\phi_{\mathbf{t}}(P)$, respectively; we may assume that these coordinates all lie in $L$.

Then $[x_i y_j]_{0\le i\le n,\ 0\le j\le m}$ is a system of homogeneous coordinates for $\phi_{\mathbf{s}\otimes\mathbf{t}}$, and we have

$$\begin{aligned} H_L(\phi_{\mathbf{s}\otimes\mathbf{t}}(P)) &= \prod_{v\in M_L} \max_{i,j} \|x_i y_j\|_v \\ &= \prod_{v\in M_L} \left(\max_i \|x_i\|_v\right)\left(\max_j \|y_j\|_v\right) \\ &= H_L(\phi_{\mathbf{s}}(P))\cdot H_L(\phi_{\mathbf{t}}(P)). \end{aligned}$$

Switching to logarithmic heights then gives (5.5.1). □

**Lemma 5.6.** *Let $X$ be a projective scheme over $k$ and let $\mathscr{L}$ be a line sheaf on $X$. Then there exist globally generated line sheaves $\mathscr{M}_1$ and $\mathscr{M}_2$ on $X$ such that $\mathscr{L}\cong\mathscr{M}_1\otimes\mathscr{M}_2^\vee$.*

*Proof.* Let $\mathscr{M}$ be an ample line sheaf on $X$. By the definition of ampleness,

$$\mathscr{M}_1 = \mathscr{L}\otimes\mathscr{M}^{\otimes n} \qquad \text{and} \qquad \mathscr{M}_2 = \mathscr{M}^{\otimes n}$$

are globally generated for all sufficiently large integers $n$. For any such $n$, $\mathscr{M}_1$ and $\mathscr{M}_2$ satisfy the given conditions. □

We are now prepared to define the height on a projective scheme.

**Definition 5.7.** Let $X$ be a projective scheme over $k$ and let $\mathscr{L}$ be a line sheaf on $X$. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be globally generated line sheaves on $X$ such that $\mathscr{L}\cong\mathscr{M}_1\otimes\mathscr{M}_2^\vee$ (Lemma 5.6). Let $\mathbf{s}=(s_0,\dots,s_n)$ and $\mathbf{t}=(t_0,\dots,t_n)$ be systems of global sections which generate $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. Then we say that a function $h_{\mathscr{L},k}\colon X(\bar{k})\to\mathbb{R}$ is a **height function** on $X$ relative to $\mathscr{L}$ and $k$ if

$$h_{\mathscr{L},k} = h_{\mathbf{s},k} - h_{\mathbf{t},k} + O(1).$$

By Lemma 5.4, this definition does not depend on the choice of $\mathbf{s}$ and $\mathbf{t}$; by Lemma 5.5, it does not depend on the choices of $\mathscr{M}_1$ and $\mathscr{M}_2$.

*For the rest of this section, we say that two real-valued functions are* **equivalent** *if their difference is bounded.*

Thus the notion of height function on $X$ relative to $\mathscr{L}$ and $k$ determines a well-defined equivalence class of functions $h_{\mathscr{L},k}\colon X(\bar{k})\to\mathbb{R}$.

Often we will shorten the notation and say "let $h_{\mathscr{L},k}$ be a height function" in place of "let $h_{\mathscr{L},k}$ be a height function on $X$ relative to $\mathscr{L}$ and $k$."

**Lemma 5.8.** *Let $X$ be a projective scheme over $k$.*

(a). *Let $\mathscr{L}$ be the trivial line sheaf $\mathscr{O}_X$ on $X$. Then $\mathbf{s}=(s_0)$, where $s_0=1$, is a system of global sections which generates $\mathscr{L}$, and in this case*

$$h_{\mathbf{s},k} = 0.$$

(b). *Let* $\mathscr{L}$ *be a globally generated line sheaf on* $X$*, let* $\mathbf{s} = (s_0, \dots, s_n)$ *be a system of global sections which generates* $\mathscr{L}$*, and let* $h_{\mathscr{L},k}$ *be a height function. Then*

$$h_{\mathscr{L},k} = h_{\mathbf{s},k} + O(1).$$

*Proof.* Part (a) follows from the the fact that $\mathbb{P}^0_k$ consists of a single point, with homogeneous coordinate $[1]$. Part (b) follows by taking $\mathscr{M}_1 = \mathscr{L}$ and $\mathscr{M}_2 = \mathscr{O}_X$ in Definition 5.7. □

**Lemma 5.9.** *Let* $\psi\colon X \to Y$ *be a morphism of projective schemes over* $k$*, and let* $\mathscr{L}$ *be a line sheaf on* $Y$*. Let* $h_{\mathscr{L},k}$ *and* $h_{\psi^*\mathscr{L},k}$ *be height functions on* $Y$ *and* $X$ *relative to* $\mathscr{L}$ *and* $\phi^*\mathscr{L}$*, respectively. Then*

$$h_{\psi^*\mathscr{L},k}(P) = h_{\mathscr{L},k}(\psi(P)) + O(1)$$

*for all* $P \in X(\bar{k})$*.*

*Proof.* By Lemma 5.6 and linearity, we may assume that $\mathscr{L}$ is globally generated. Let $\mathbf{s} = (s_0, \dots, s_n)$ be a system of global sections which generates $\mathscr{L}$. Then

$$\psi^*\mathbf{s} := (\psi^* s_0, \dots, \psi^* s_n)$$

is a system of global sections which generates $\psi^*\mathscr{L}$. Hence by Lemma 5.8b we may assume that $h_{\mathscr{L},k} = h_{\mathbf{s},k}$ and that $h_{\phi^*\mathscr{L},k} = h_{\phi^*\mathbf{s},k}$. Let $\phi_{\mathbf{s}}\colon Y \to \mathbb{P}^n_k$ and $\phi_{\psi^*\mathbf{s}}\colon X \to \mathbb{P}^n_k$ be the maps associated to $\mathbf{s}$ and $\psi^*\mathbf{s}$, respectively, by Lemma 5.2. Then the lemma follows from the fact that $\phi_{\psi^*\mathbf{s}} = \phi_{\mathbf{s}} \circ \psi$. □

**Lemma 5.10.** *Let* $h_{\mathscr{O}(1),k}$ *be a height function on* $\mathbb{P}^n_k$ *relative to the standard twisting sheaf* $\mathscr{O}(1)$*. Then*

$$h_{\mathscr{O}(1),k}(P) = h_k(P) + O(1)$$

*for all* $P \in \mathbb{P}^n_k(\bar{k})$*.*

*Proof.* Indeed, $\mathbf{x} = (x_0, \dots, x_n)$ is a system of global sections which generates $\mathscr{O}(1)$, and $\phi_{\mathbf{x}}\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ (as defined by Lemma 5.2) is the identity morphism. □

The properties of heights proved in this section can be summarized as follows:

**Theorem 5.11** (The "Height Machine"). *Let* $\mathscr{C}$ *be the category of projective schemes over* $k$*. Then there is a unique function*

$$(X, \mathscr{L}) \mapsto [h_{\mathscr{L},k}],$$

*taking pairs* $(X, \mathscr{L})$*, where* $X \in \mathscr{C}$ *and* $\mathscr{L} \in \operatorname{Pic}(X)$*, to equivalence classes* $[h_{\mathscr{L},k}]$ *of height functions* $h_{\mathscr{L},k}\colon X(\bar{k}) \to \mathbb{R}$ *relative to* $\mathscr{L}$ *and* $k$*, such that:*

(i). *(Additivity) For each* $X$ *the map* $\mathscr{L} \mapsto [h_{\mathscr{L},k}]$ *is a group homomorphism from the group* $\operatorname{Pic}(X)$ *to the additive group of equivalence classes of real-valued functions on* $X(\bar{k})$*.*

(ii). *(Functoriality) For each morphism* $\phi\colon X \to Y$ *in* $\mathscr{C}$ *and each line sheaf* $\mathscr{L}$ *on* $Y$,

$$[h_{\phi^*\mathscr{L},k}] = [h_{\mathscr{L},k} \circ \phi].$$

(iii). *(Normalization) If* $X = \mathbb{P}^n_k$ *for some* $n \in \mathbb{N}$, *then*

$$[h_{\mathscr{O}(1),k}] = [h_k],$$

*(where the function* $h_k$ *on the right-hand side is the logarithmic height on* $\mathbb{P}^n_k$ *defined in Section 3).*

*Proof.* Existence follows by Definition 5.7 and Lemmas 5.9 and 5.10. Uniqueness is the only new assertion in this theorem. Let $X \in \mathscr{C}$ and let $\mathscr{L} \in \operatorname{Pic}(X)$. By additivity, it suffices to prove uniqueness in the case where $\mathscr{L}$ is globally generated. Let $\mathbf{s} = (s_0, \dots, s_n)$ be a system of global sections which generates $\mathscr{L}$. By applying the functoriality condition to the morphism $\phi_{\mathbf{s}}\colon X \to \mathbb{P}^n_k$ associated to $\mathbf{s}$ by Lemma 5.2, it suffices to prove uniqueness in the case where $X = \mathbb{P}^n$ and $\mathscr{L} = \mathscr{O}(1)$. But this is immediate by the normalization condition. □

We now extend the definition of heights to proper schemes.

**Definition 5.12.** Let $X$ be a proper non-projective scheme over $k$ and let $\mathscr{L}$ be a line sheaf on $X$. Then a function $h_{\mathscr{L},k}\colon X(\bar{k}) \to \mathbb{R}$ is a **height function** for $\mathscr{L}$ and $k$ if there exists a projective scheme $Y$ over $k$, a surjective morphism $\phi\colon Y \to X$, a section $\sigma\colon X(\bar{k}) \to Y(\bar{k})$ of the function $Y(\bar{k}) \to X(\bar{k})$ induced by $\phi$, and a height function $h_{\phi^*\mathscr{L},k}$ on $Y$ relative to $\phi^*\mathscr{L}$, such that

$$h_{\mathscr{L},k}(P) = h_{\phi^*\mathscr{L},k}(\sigma(P)) + O(1) \tag{5.12.1}$$

for all $P \in X(\bar{k})$.

This definition is independent of the choice of $h_{\phi^*\mathscr{L}}$, by Theorem 5.11. We next show that it is independent of the choices of $\phi$ and $\sigma$, and that it is compatible with the definition of height function when $X$ is projective.

**Lemma 5.13.** *Let* $f\colon X_1 \to X_2$ *be a morphism of proper schemes over* $k$. *For* $i = 1, 2$ *let* $Y_i$ *be a projective scheme over* $k$, *let* $\phi_i\colon Y_i \to X_i$ *be a surjective morphism, and let* $\sigma_i\colon X_i(\bar{k}) \to Y_i(\bar{k})$ *be a section of the function* $Y_i(\bar{k}) \to X_i(\bar{k})$ *induced by* $\phi_i$. *Let* $\mathscr{L}$ *be a line sheaf on* $X_2$. *Let* $h_{\phi_1^* f^*\mathscr{L},k}$ *and* $h_{\phi_2^*\mathscr{L},k}$ *be height functions on* $Y_1$ *and* $Y_2$ *relative to* $\phi_1^* f^*\mathscr{L}$ *and* $\phi_2^*\mathscr{L}$, *respectively. Then*

$$h_{\phi_1^* f^*\mathscr{L},k}(\sigma_1(P)) = h_{\phi_2^*\mathscr{L},k}(\sigma_2(f(P))) + O(1)$$

*for all* $P \in X_1(\bar{k})$.

*Proof.* Let $Y = Y_1 \times_{X_2} Y_2$, with projection maps $\mathrm{pr}_i\colon Y \to Y_i$ ($i = 1, 2$) and structural morphism $\phi\colon Y \to X_2$. Then $Y$ is projective over $k$; let $h_{\phi^*\mathscr{L},k}$ be a height

function on $Y$ relative to $\phi^*\mathscr{L}$ and $k$. Also choose $\sigma\colon X_1(\bar{k}) \to Y(\bar{k})$ such that $\mathrm{pr}_1(\sigma(P)) = \sigma_1(P)$ and $\mathrm{pr}_2(\sigma(P)) = \sigma_2(f(P))$ for all $P \in X_1(\bar{k})$. Then, since

$$\phi^*\mathscr{L} \cong \mathrm{pr}_1^*\,\phi_1^* f^*\mathscr{L} \cong \mathrm{pr}_2^*\,\phi_2^*\mathscr{L},$$

we have

$$h_{\phi_1^* f^*\mathscr{L},k}(\sigma_1(P)) = h_{\phi^*\mathscr{L},k}(\sigma(P)) + O(1) = h_{\phi_2^*\mathscr{L},k}(\sigma_2(f(P))) + O(1). \tag{5.13.1}$$ □

**Remark.** In the notation of the above proof, $Y$ may be reducible even if $Y_1$ and $Y_2$ are irreducible. Otherwise, everything in this chapter could have been stated in terms of varieties instead of schemes. The reader who prefers to work in the category of varieties can look at $Z_{\mathrm{red}}$ for each irreducible component $Z$ of $Y$ to obtain (5.13.1).

**Proposition 5.14.** *Let $X$ be a proper scheme over $k$, let $\mathscr{L}$ be a line sheaf on $X$, and let $h_{\mathscr{L},k}$ be a height function for $\mathscr{L}$ and $k$. Let $Y$ be a projective scheme over $k$, let $\phi\colon Y \to X$ be a surjective morphism, let $\sigma\colon X(\bar{k}) \to Y(\bar{k})$ be a section of the function $Y(\bar{k}) \to X(\bar{k})$ induced by $\phi$, and let $h_{\phi^*\mathscr{L},k}$ be a height function on $Y$ relative to $\phi^*\mathscr{L}$. Then*

$$h_{\mathscr{L},k}(P) = h_{\phi^*\mathscr{L},k}(\sigma(P)) + O(1)$$

*for all $P \in X(\bar{k})$.*

*Proof.* Let $Y'$ be a projective scheme over $k$ such that there exists a surjective morphism $\phi'\colon Y \to X$, and let $\sigma'\colon X(\bar{k}) \to Y'(\bar{k})$ be a section of the function $Y'(\bar{k}) \to X(\bar{k})$ induced by $\phi'$. The proposition then follows by applying Lemma 5.13 with $X_1 = X_2 = X$, $f = \mathrm{Id}_X$, $Y_1 = Y$, $\phi_1 = \phi$, $\sigma_1 = \sigma$, $Y_2 = Y'$, $\phi_2 = \phi'$, and $\sigma_2 = \sigma'$. (It is also immediate when $X$ is projective.) □

**Remark 5.15.** The restriction of $\mathscr{L}$ to fibers of $\pi$ is trivial, so the height $h_{\phi^*\mathscr{L},k}$ will be bounded on such fibers. Therefore $h_{\phi^*\mathscr{L},k}(\sigma(P))$ will be independent of the choice of $\sigma$ up to $O(1)$, but the implied constant might (a priori) depend on $P$. The above proposition shows that, in fact, the constant is uniform as $P$ varies.

**Lemma 5.16.** *Let $f\colon X_1 \to X_2$ be a morphism of proper schemes over $k$, let $\mathscr{L}$ be a line sheaf on $X_2$, and let $h_{f^*\mathscr{L},k}$ and $h_{\mathscr{L},k}$ be height functions on $X_1$ and $X_2$ relative to $f^*\mathscr{L}$ and $\mathscr{L}$, respectively. Then $h_{f^*\mathscr{L},k}(P) = h_{\mathscr{L},k}(f(P))$ for all $P \in X_1(\bar{k})$.*

*Proof.* This follows from Proposition 5.14 (if $X_1$ or $X_2$ is projective) and Lemma 5.13. □

**Theorem 5.17.** *Theorem 5.11 (the "height machine") remains true when $\mathscr{C}$ is replaced by the category of proper schemes over $k$.*

*Proof.* First of all, given a proper scheme $X$ and a line sheaf $\mathscr{L}$ on $X$, there exists a projective scheme $Y$ and a surjective morphism $\phi\colon Y \to X$ by Chow's lemma, ([**EGA**],

II 5.6.1) or ([**H 2**], II Ex. 4.10). Hence, for all such $X$ and $\mathscr{L}$, a height function $h_{\mathscr{L},k}$ exists by Proposition 5.14 and existence of heights on projective schemes. Additivity of the new $h_{\mathscr{L},k}$ follows immediately from additivity of $h_{\mathscr{L},k}$ on projective schemes. Functoriality holds by Lemma 5.16, and the normalization condition holds by Theorem 5.11. Finally, uniqueness follows by Proposition 5.14 and the uniqueness assertion of Theorem 5.11. □

The notion of heights cannot be extended further to, *e.g.*, affine varieties. For example, on $\mathbb{A}^1_k$ all line sheaves are trivial, but not all heights are bounded.

Finally, we note that it is often convenient to refer to heights relative to a divisor instead of a line sheaf.

**Definition 5.18.** Let $X$ be proper over $k$ and let $D$ be a Cartier divisor on $X$. Then a **height function** $h_{D,k}\colon X(\bar{k}) \to \mathbb{R}$ relative to $D$ and $k$ is a height function relative to $\mathscr{O}(D)$ and $k$.

The following result is then immediate.

**Proposition 5.19.** *Let $D_1$ and $D_2$ be Cartier divisors on a proper scheme $X$. Let $h_{D_1,k}$ and $h_{D_2,k}$ be height functions relative to $D_1$ and $D_2$, respectively. If $D_1$ and $D_2$ are linearly equivalent, then $h_{D_1,k} = h_{D_2,k} + O(1)$.*

*Proof.* Indeed, $\mathscr{O}(D_1) \cong \mathscr{O}(D_2)$. □

## §6. Northcott's finiteness theorem over function fields

This section gives Northcott's theorem over function fields. The method of proof is completely different from the proof for number fields.

In function fields the situation is a bit different from number fields, in that if the constant field $F$ is infinite, then the set in question may also be infinite. For example, let $F$ be an infinite field; then the set $F$ of constants is an infinite set of rational points whose (logarithmic) height is $0$. A set of points of bounded degree and height is, however, described as in one of your homework problems.

## §7. Other properties of heights

In this section we prove additional properties of height functions. Mostly these concern heights relative to ample and big divisors.

*Throughout this section, $X$ is a proper scheme over $k$ and $\mathscr{L}$ is a line sheaf on $X$.*

**Proposition 7.1.** *Let $L$ be a finite extension of $k$. Let $X' = X \times_k L$, let $\phi\colon X' \to X$ be the canonical projection, and let $\mathscr{L}' = \phi^*\mathscr{L}$. Let $h_{\mathscr{L},k}$ and $h_{\mathscr{L}',L}$ be height functions on $X$ and $X'$ relative to $\mathscr{L}$ and $k$, and to $\mathscr{L}'$ and $L$, respectively. Then*

$$h_{\mathscr{L}',L}(P) = [L:k]h_{\mathscr{L},k}(\phi(P)) + O(1)$$

*for all $P \in X'(\bar{L})$.*

*Proof.* By functoriality and additivity, we may assume that $X = \mathbb{P}^n_k$ and $\mathscr{L} = \mathscr{O}(1)$. Then the result follows from the corresponding result for heights on projective spaces, Lemma 3.2b. □

**Theorem 7.2** (Northcott). *Suppose that $X$ is projective and that $\mathscr{L}$ is ample. Let $h_{\mathscr{L},k}$ be a height function, let $d \in \mathbb{Z}_{>0}$, and let $C \in \mathbb{R}$. Then the set*

$$\{P \in X(\bar{k}) \mid [k(P) : k] \le d \text{ and } h_{\mathscr{L},k}(P) \le C\}$$

*is finite if $k$ is a number field, or is composed of an algebraic family if $k$ is a function field.*

*Proof.* First assume that $k$ is a number field. After replacing $\mathscr{L}$ with $\mathscr{L}^{\otimes n}$ and $C$ with $nC$ for some $n \in \mathbb{Z}_{>0}$, we may assume that $\mathscr{L}$ is very ample. Then, by functoriality, we may assume that $X$ is a closed subscheme of $\mathbb{P}^n_k$ for some $n$; in that case the result follows from Northcott's theorem for projective space, Theorem 3.4.

The function field case follows from Corollary 6.4. □

**Theorem 7.3.** *Let $h_{\mathscr{L},k}$ be a height function on $X$, and let $U$ be an open subset of $X$ such that $\mathscr{L}$ is globally generated over $U$. Then*

$$h_{\mathscr{L},k}(P) \ge O(1) \qquad \text{for all } P \in U(\bar{k}).$$

*Proof.* We may assume that $U \ne \emptyset$. By Chow's lemma and functoriality we may assume that $X$ is projective. Let $\mathbf{u} = (u_0, \dots, u_\ell)$ be a system of global sections of $\mathscr{L}$ which generate $\mathscr{L}$ over $U$. Let $\mathscr{M}$ be a line sheaf on $X$ such that $\mathscr{M}$ and $\mathscr{L} \otimes \mathscr{M}$ are globally generated. Let $\mathbf{s} = (s_0, \dots, s_n)$ and $\mathbf{t} = (t_0, \dots, t_n)$ be systems of global sections which generate $\mathscr{M}$ and $\mathscr{L} \otimes \mathscr{M}$, respectively. We may assume that the first $(l+1)(n+1)$ coordinates of $\mathbf{t}$ are $u_j \otimes s_i$ ($0 \le i \le n$; $0 \le j \le \ell$). Since

$$h_{\mathscr{L},k} = h_{\mathbf{t},k} - h_{\mathbf{s},k} + O(1),$$

it will suffice to prove that $h_{\mathbf{t},k}(P) \ge h_{\mathbf{s},k}(P)$ for all $P \in U(\bar{k})$. Let $\phi_{\mathbf{u}} \colon U \to \mathbb{P}^\ell_k$, $\phi_{\mathbf{s}} \colon X \to \mathbb{P}^n_k$, and $\phi_{\mathbf{t}} \colon X \to \mathbb{P}^m_k$ be the morphisms defined by $\mathbf{u}$, $\mathbf{s}$, and $\mathbf{t}$, respectively, by Lemma 5.2. Pick $P \in U(\bar{k})$, let $L = k(P)$, and let $[w_0 : \cdots : w_\ell]$ and $[x_0 : \cdots : x_n]$ be homogeneous coordinates for $\phi_{\mathbf{u}}(P)$ and $\phi_{\mathbf{s}}(P)$, respectively. We may assume that these coordinates all lie in $L$. Then there exist homogeneous coordinates $[y_0 : \cdots : y_m]$ for $\phi_{\mathbf{t}}(P)$ such that the first $(\ell+1)(n+1)$ coordinates are $w_j x_i$

$(0 \le i \le n;\ 0 \le j \le \ell)$. Fix $j_P \in \{0, \dots, \ell\}$ such that $w_{j_P} \neq 0$. Then we have

$$\begin{aligned} H_L(\phi_{\mathbf{t}}(P)) &= \prod_{v \in M_L} \max_{0 \le i \le m} \|y_i\|_v \\ &\ge \prod_{v \in M_L} \max_{0 \le i < (\ell+1)(n+1)} \|y_i\|_v \\ &= \prod_{v \in M_L} \max_{\substack{0 \le j \le \ell \\ 0 \le i \le n}} \|w_j x_i\|_v \\ &\ge \prod_{v \in M_L} \|w_{j_P}\|_v \cdot \prod_{v \in M_L} \max_{0 \le i \le n} \|x_i\|_v \\ &= 1 \cdot H_L(\phi_{\mathbf{s}}(P)) \end{aligned}$$

by the product formula. Taking logarithms and normalizing gives $h_{\mathbf{t},k}(P) \ge h_{\mathbf{s},k}(P)$, which implies the result. □

**Corollary 7.4.** *If $\mathscr{L}$ is globally generated, then any height function $h_{\mathscr{L},k}$ is bounded from below.*

**Corollary 7.5.** *If $\mathscr{L}$ is ample, then any height function $h_{\mathscr{L},k}$ is bounded from below.*

*Proof.* By additivity, we may replace $\mathscr{L}$ with $\mathscr{L}^{\otimes n}$ for some positive integer $n$ such that $\mathscr{L}^{\otimes n}$ is very ample. Then, *a fortiori,* $\mathscr{L}$ is globally generated. □

The following example shows what can happen in the context of Theorem 7.3.

**Example 7.6.** Let $\pi\colon X \to \mathbb{P}^2_k$ be the blowing-up of $\mathbb{P}^2_k$ at a rational point $P_0$, let $E$ be the exceptional divisor of the blowing-up, and let $\mathscr{L} = \pi^*\mathscr{O}(1) \otimes \mathscr{O}(E)$. Then $E \cong \mathbb{P}^1_k$, and $\mathscr{L}\big|_E \cong \mathscr{O}(-1)$ via this isomorphism. Every global section $s$ of $\mathscr{L}$ must vanish on $E$, because

$$s\big|_E \in \Gamma\big(E, \mathscr{L}\big|_E\big) = 0.$$

On the other hand, every global section of $\mathscr{O}(1)$ on $\mathbb{P}^2_k$ pulls back to give a global section of $\pi^*\mathscr{O}(1)$; this in turn gives a global section of $\mathscr{L}$. Thus $\mathscr{L}$ is globally generated on $U := X \setminus E$, but not on any larger open set. If $h_{\mathscr{L}}$ is a height function, then by Theorem 7.3,

$$h_{\mathscr{L},k}(P) \ge O(1) \qquad \text{for all } P \in U(\bar{k}).$$

But since $\mathscr{L}\big|_E \cong \mathscr{O}(-1)$, the function $h_{\mathscr{L},k}$ is bounded from above but not from below on $E(\bar{k})$.

By Corollary 7.5, the height relative to an ample line sheaf is bounded from below. The following proposition shows that, in fact, the height relative to an ample line sheaf is the largest possible height, up to a multiplicative constant.

**Proposition 7.7.** *Suppose that $X$ is projective, let $\mathscr{L}$ be a line sheaf on $X$, and let $\mathscr{M}$ be an ample line sheaf on $X$. Let $h_{\mathscr{L},k}$ and $h_{\mathscr{M},k}$ be height functions. Then there is a constant $C$ such that*

$$h_{\mathscr{L},k} \le C h_{\mathscr{M},k} + O(1).$$

*Proof.* For $n$ large enough, $\mathscr{L}^\vee \otimes \mathscr{M}^{\otimes n}$ is globally generated. Therefore, by Corollary 7.4, a corresponding height function satisfies

$$h_{\mathscr{L}^\vee \otimes \mathscr{M}^{\otimes n},k} \ge O(1).$$

The result then follows by additivity of the height function. □

Proposition 5.19 showed how heights behave with respect to linear equivalence. We now show how they behave with respect to numerical equivalence (and therefore algebraic equivalence).

**Proposition 7.8.** *Suppose that $X$ is projective, let $\mathscr{L}_1$ and $\mathscr{L}_2$ be numerically equivalent line sheaves on $X$, and let $\mathscr{M}$ be an ample line sheaf on $X$. Let $h_{\mathscr{L}_1,k}$, $h_{\mathscr{L}_2,k}$, and $h_{\mathscr{M},k}$ be height functions. Then for all $\epsilon > 0$,*

$$|h_{\mathscr{L}_1,k} - h_{\mathscr{L}_2,k}| \le \epsilon h_{\mathscr{M},k} + O(1).$$

*Proof.* Let $n$ be a positive integer with $1/n < \epsilon$. By ([**H 1**], Ch. I, Cor. 7.2), ampleness of a line sheaf depends only on its numerical equivalence class. In the situation at hand, this implies that $\mathscr{M} \otimes (\mathscr{L}_1^\vee \otimes \mathscr{L}_2)^{\otimes n}$ is ample. Thus, by Corollary 7.5, a corresponding height function satisfies

$$h_{\mathscr{M} \otimes (\mathscr{L}_1^\vee \otimes \mathscr{L}_2)^{\otimes n},k} \ge O(1).$$

By the additivity property of heights, we then have

$$h_{\mathscr{L}_1,k} \le h_{\mathscr{L}_2,k} + \frac{1}{n} h_{\mathscr{M},k} + O(1).$$

The proposition then follows by symmetry. □

By Proposition 7.7, this condition is independent of the choice of ample line sheaf $\mathscr{M}$. Sharper bounds are possible: see ——— (include an example showing that the square root is best possible).

Finally, we consider an analogue of Proposition 7.7 for big line sheaves.

**Proposition 7.9.** *Let $X$ be a complete variety over $k$, let $\mathscr{L}$ be a line sheaf on $X$, and let $\mathscr{M}$ be a big line sheaf on $X$. Let $h_{\mathscr{L},k}$ and $h_{\mathscr{M},k}$ be height functions. Then there exists a constant $C$ and a proper Zariski-closed subset $Z$ of $X$ such that*

$$h_{\mathscr{L},k}(P) \le C h_{\mathscr{M},k}(P) + O(1) \qquad \text{for all } P \in (X \setminus Z)(\bar{k}).$$

*Moreover, the set $Z$ does not depend on $\mathscr{L}$.*

*Proof.* By Chow's lemma and functoriality of the height, we may assume that $X$ is projective. By Kodaira's lemma (Theorem **0:**8.6), there is a positive integer $n$, an ample line sheaf $\mathscr{M}'$, and an effective Cartier divisor $D$ such that

$$\mathscr{M}^{\otimes n} \cong \mathscr{M}' \otimes \mathscr{O}(D).$$

Then, letting $h_{\mathscr{M}',k}$ and $h_{D,k}$ be height functions and letting $Z = \operatorname{Supp} D$, we have

$$\begin{aligned} h_{\mathscr{M}',k}(P) &= nh_{\mathscr{M},k}(P) - h_{D,k}(P) + O(1) \\ &\le nh_{\mathscr{M},k}(P) + O(1) \end{aligned}$$

for all $P \in (X \setminus Z)(\bar{k})$, since $\mathscr{O}(D)$ is globally generated on $X \setminus Z$. The proposition then follows by Proposition 7.7. □

Thus, on a projective variety, an ample divisor gives the largest possible height, up to a constant factor and an additive constant. On a complete non-projective variety there is no ample line sheaf (perforce), and in fact there may not even be a nontrivial line sheaf ([**F**], pp. 25–26 and p. 72). If a big line sheaf exists, though, then it would give a largest possible height (as above) outside of a proper Zariski-closed subset.
——— Other topics: heights & families (Silverman, Call, de Diego); asymptotics of heights (Thunder; Manin-etc.).

## References

[F] W. Fulton, *Introduction to toric varieties*, Annals of Math. Studies 131, Princeton Univ. Press, Princeton, NJ, 1993.

[H 1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lect. Notes Math. 156, Springer-Verlag, Berlin etc., 1970.

[H 2] ———, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.

[L] S. Lang, *Algebra*, 3rd edition, Addison-Wesley, Reading, Mass., 1993.

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