CHAPTER 0
PRELIMINARY MATERIAL

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This chapter gives some preliminary material on number theory and algebraic geometry.

Section 1 gives basic preliminary notation, both mathematical and logistical. Section 2 describes what algebraic geometry is assumed of the reader, and gives a few conventions that will be assumed here. Sections 3 and 4 give a few more details on schematic denseness, associated points, rational maps, rational sections of line sheaves, and how the above relate to the relation between Cartier divisors and line sheaves. Sections 5 and 6 discuss the field of definition of a closed subscheme and of a variety, respectively. Section 7 sets the basic notation and gives some fundamental results in number theory.

The remaining sections of this chapter describe some more specialized topics in algebraic geometry that will prove useful later: Kodaira's lemma in Section 8, and descent in Section 9.

§1. General notation

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the ring of rational integers and the fields of rational numbers, real numbers, and complex numbers, respectively. The symbol $\mathbb{N}$ signifies the natural numbers, which in this book start at zero: $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. When it is necessary to refer to the positive integers, we use subscripts: $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$. Similarly, $\mathbb{R}_{\ge 0}$ stands for the set of nonnegative real numbers, etc.

The set of **extended real numbers** is the set $\overline{\mathbb{R}} := \{-\infty\} \coprod \mathbb{R} \coprod \{\infty\}$. It carries the obvious ordering.

If k is a field, then $\bar{k}$ denotes an algebraic closure of k. If $\alpha \in \bar{k}$, then $\mathrm{Irr}_{\alpha,k}(X)$ is the (unique) monic irreducible polynomial $f \in k[X]$ for which $f(\alpha) = 0$.

Unless otherwise specified, the wording **almost all** will mean all but finitely many.

Numbers (such as Section 2, Theorem 2.3, or (2.3.5)) refer to the chapter in which they occur, unless they are preceded by a number or letter and a colon in bold-face type (*e.g.*, Section **3:**2, Theorem **A:**2.5, or (**7:**2.3.5)), in which case they refer to the chapter or appendix indicated by the bold-face number or letter, respectively.

§2. Conventions and basic results in algebraic geometry

It is assumed that the reader is familiar with the basics of algebraic geometry as given, *e.g.*, in the first three chapters of [**H 2**], especially the first two. Note, however, that some conventions are different here.

This book will primarily use the language of schemes, rather than of varieties. The reader who prefers the more elementary approach of varieties, however, will often be able to mentally substitute the word *variety* for *scheme* without much loss, especially in the first few chapters.

With the exception of Appendix B, all schemes are assumed to be separated. Usually, schemes will also be assumed to be of finite type over some base scheme.

We often omit *Spec* when it is clear from the context; *e.g.*, $X(A)$ means $X(\operatorname{Spec} A)$ when A is a ring, $\mathbb{P}^n_A$ means $\mathbb{P}^n_{\operatorname{Spec} A}$, and $X \times_A B$ means $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ when A and B are rings.

The following definition gives slightly different names for some standard objects.

Definition 2.1. Let X be a scheme. Then a **vector sheaf** on X is a locally free sheaf of $\mathscr{O}_X$-modules on X of constant finite rank. Its **rank** is the value of that constant rank (if $X \neq \emptyset$). A **morphism** of vector sheaves is a morphism of $\mathscr{O}_X$-modules on X. A **line sheaf** is a vector sheaf of rank 1.

A vector sheaf $\mathscr{F}$ of rank r on X can be described concretely by giving an open cover $\{U_i\}_{i\in I}$ of X and isomorphisms $\phi_i\colon \mathscr{F}\big|_{U_i} \xrightarrow{\sim} \mathscr{O}^r_{U_i}$ for all $i \in I$ such that for all $i, j \in I$, the automorphism

$$\phi_i\big|_{U_i\cap U_j} \circ \phi_j^{-1}\big|_{U_i\cap U_j} \colon \mathscr{O}^r_{U_i\cap U_j} \to \mathscr{O}^r_{U_i\cap U_j}$$

is given by an element of $\mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathscr{O}_X))$. If $\mathscr{G}$ is another vector sheaf on X, with corresponding open cover $\{V_j\}_{j\in J}$ and isomorphisms $\psi_j\colon \mathscr{G}\big|_{V_j} \xrightarrow{\sim} \mathscr{O}^s_{V_j}$, then a morphism of vector sheaves $\mathscr{F} \to \mathscr{G}$ is a morphism $\rho\colon \mathscr{F} \to \mathscr{G}$ of sheaves such that, for all $i \in I$ and all $j \in J$, the morphism $\mathscr{O}^r_{U_i\cap V_j} \to \mathscr{O}^s_{U_i\cap V_j}$ corresponding to $\rho\big|_{U_i\cap V_j}$ via the isomorphisms ϕ_i and ψ_j is a linear homomorphism of $\mathscr{O}_{U_i\cap V_j}$-modules.

A line sheaf is also called an invertible sheaf by many authors.

Varieties

Not all authors use the same definition of variety. Here we use the following definition.

Definition 2.2. Let k be a field. A **variety over** k, also called a k-variety, is an integral scheme, of finite type (and separated) over $\operatorname{Spec} k$. If it is also proper over k, then we say it is **complete**. A **curve** is a variety of dimension 1. A **morphism** of k-varieties is a morphism of schemes over $\operatorname{Spec} k$. A **subvariety** (resp. **open subvariety**, **closed subvariety**) of a given variety over k is an integral subscheme (resp. open integral subscheme, closed integral subscheme) of that variety (with induced map to $\operatorname{Spec} k$).

Note that, since k is not assumed to be algebraically closed, the set of closed points of a variety X over k is the set $X(\bar{k})$, modulo the action of $\mathrm{Aut}_k(\bar{k})$. Also, the residue field $k(P)$ for a closed point P will in general be a finite extension of k.

Remark 2.3. We have not assumed a variety to be geometrically integral. An advantage of this choice is that every irreducible closed subset of a variety will again be a variety, so that there is a natural one-to-one correspondence between the set of points of a variety and its set of subvarieties. This choice also agrees with the general philosophy that definitions should be weak. Since integrality is an absolute definition, if X is a variety over a number field k, then it is also a variety over any smaller number field k', since one can compose $X \to \operatorname{Spec} k$ with the map $\operatorname{Spec} k \to \operatorname{Spec} k'$ to get X to be a scheme over k'. The latter is discussed more in Section 6.

In addition, we have (not just for regular field extensions):

Proposition 2.4. *The association $X \mapsto K(X)$ induces an arrow-reversing equivalence of categories between the category of varieties and dominant rational maps over k, and the category of finitely-generated field extensions of k.*

Proof. The proof of ([**H 2**], I Thm. 4.4) extends directly to the present case; see also ([**EGA**], I 7.1.16). □

Remark 2.5. Let $k \subseteq L$ be fields. Then, for varieties (or schemes) X over k, the sets $X(L)$ satisfy the following basic properties:

(a). $\mathbb{A}^n(L) = L^n$ in the obvious way. A similar statement holds for $\mathbb{P}^n(L)$.
(b). A morphism $f\colon X_1 \to X_2$ of schemes over k induces a natural map

$$f_L\colon X_1(L) \to X_2(L)\ .$$

(c). If f is an immersion, then f_L is injective.
(d). If f is surjective and L is algebraically closed, then f_L is surjective.
(e). Since the field L has no nilpotents, $X_{\mathrm{red}}(L) = X(L)$.
(f). An inclusion $L_1 \subseteq L_2$ of fields induces a natural injection $X(L_1) \hookrightarrow X(L_2)$.
(g). If k' is an extension field of k such that both k and L are contained in some larger field, then for schemes X over k there is a natural injection $X(L) \hookrightarrow (X \times_k k')(k'L)$.

Note, however, that if L is not algebraically closed, then f_L in part (d) need not be surjective; consider for example $f\colon \mathbb{A}^1_{\mathbb{Q}} \to \mathbb{A}^1_{\mathbb{Q}}$ defined by $z \mapsto z^2$.

We note that cohomology is geometric in nature: Let X be a scheme over k, let $\mathscr{F}$ be a quasi-coherent sheaf on X, let L be a field containing k, let $X_L = X \times_k L$, let $f\colon X_L \to X$ be the projection morphism, and let $\mathscr{F}_L = f^*\mathscr{F}$. Then, by the fact that cohomology commutes with flat base extension ([**H 2**], III Prop. 9.3),

$$H^i(X_L, \mathscr{F}_L) \cong H^i(X, \mathscr{F}) \otimes_k L$$

for all $i \geq 0$.

Later chapters of the book will work extensively with schemes of finite type over the ring of integers of a number field. Such schemes have many of the same properties as varieties over a field. We describe one set of such properties here, generalizing a basic result on dimension in ([**H**], I Thm. 1.8A).

Proposition 2.6. *Let $f\colon X \to Y$ be a morphism of finite type, where X and Y are noetherian integral schemes and Y is regular of dimension 1. Assume that (i) Y has infinitely many points, or (ii) f is proper. Then:*

(a). *For all closed points x of X, the point $y := f(x)$ is a closed point of Y, and the induced extension $k(x)/k(y)$ of residue fields is a finite extension.*

(b). $\dim X = \mathrm{tr.deg}\, K(X)/K(Y) + 1$*; and*

(c). *X is catenary and equicodimensional; in particular,*

$$\dim \overline{\{x\}} + \mathrm{codim}(x, X) = \dim X \tag{2.6.1}$$

for all points $x \in X$. (Here **equicodimensional** *means that all closed points of X have the same codimension in X.)*

Proof. After passing to an open affine, we may assume that Y is affine, say $Y = \mathrm{Spec}\, A$. Then A is a Dedekind domain.

We first prove (a).

Under assumption (i), A is not semilocal, hence is a Jacobson ring. The lemma then follows by the general Nullstellensatz ([**E**], Thm. 4.19).

Under assumption (ii), if x is a closed point, then so is y since by assumption f is proper, hence a closed morphism. Then x is a closed point of the fiber over y, which is a scheme of finite type over $k(y)$, and the assertion again follows from the general Nullstellensatz.

To prove (b), let $x \in X$ and let $y = f(x)$. Let $\mathrm{Spec}\, A$ and $\mathrm{Spec}\, B$ be open affine neighborhoods of x and y, respectively, with $f(\mathrm{Spec}\, B) \subseteq \mathrm{Spec}\, A$, so that B is of finite type over A. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the prime ideals of A and B corresponding to x and y, respectively. Since A is a regular ring, it is universally catenary by ([**EGA**], IV 5.6.4). Therefore B is also universally catenary, and by ([**EGA**], IV 5.6.1) we have

$$\dim A_{\mathfrak{p}} + \mathrm{tr.deg}\, B_{\mathfrak{q}}/A_{\mathfrak{p}} = \dim B_{\mathfrak{q}} + \mathrm{tr.deg}\, k(x)/k(y)\ . \tag{2.6.2}$$

Here the transcendence degree of an extension of entire rings means the transcendence degree of the corresponding extension of their fraction fields, so the second term equals $\mathrm{tr.deg}\, K(X)/K(Y)$. If x is a closed point then $\dim A_{\mathfrak{p}} = 1$ and $\mathrm{tr.deg}\, k(x)/k(y) = 0$ by part (a), so (2.6.2) becomes $\mathrm{codim}(x, X) = \mathrm{tr.deg}\, K(X)/K(Y) + 1$. Since this holds for all closed points $x \in X$, part (b) holds. It also implies that X is equicodimensional.

The above paragraph also shows that X is catenary. Finally, (2.6.1) follows from the fact that X is equicodimensional and catenary. Indeed, let

$$Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_d = X$$

be a maximal chain of irreducible subsets of X in which $\overline{\{x\}}$ occurs. We then have $d = \mathrm{codim}(Z_0, X) = \dim X$ since X is catenary and equicodimensional, respectively. (One can also obtain (2.6.1) from (2.6.2).) □

Extending ([**E**], Cor. 13.5), we then have:

Corollary 2.7. *Let $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ be morphisms as above, and let $g\colon X_1 \to X_2$ be a dominant morphism such that $f_1 = f_2 \circ g$. Then the generic fiber of g has dimension $\dim X_1 - \dim X_2$.*

§3. Associated points, rational maps, and rational sections

When dealing with schemes that are not necessarily reduced, we use a definition of rational map and rational section that is a bit different from the usual definition; see Definition 3.6. This definition is based on schematic denseness.

Definition 3.1. Let U be an open subset of a scheme X. We say that U is **schematically dense** in X if its schematic closure (the schematic image of the open immersion $U \hookrightarrow X$, or the smallest closed subscheme of X containing U) is all of X. In this situation, we also say that U is an **open dense subscheme** of X.

Often it is convenient to think of schematic denseness in terms of the associated points of X.

Proposition 3.2. *An open subset U of a noetherian scheme X is schematically dense if and only if it contains all associated points of X.*

Proof. It suffices to show this when X is an affine scheme. Let A be the affine ring of X, let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be a minimal primary decomposition of the ideal (0) in A, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ for each i. Then $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are the primes corresponding to the associated points of X.

First suppose that U contains all associated points of X. Let $\mathfrak{a}$ be the ideal corresponding to the schematic closure of U, and let $f \in \mathfrak{a}$. For all i, $\operatorname{Spec} A/\mathfrak{a}$ contains an open neighborhood of $\mathfrak{p}_i$; therefore $(A/\mathfrak{a})_{\mathfrak{p}_i/\mathfrak{a}} = A_{\mathfrak{p}_i}$. In particular, $f = 0$ in $A_{\mathfrak{p}_i}$, so $\operatorname{Ann}(f)$ meets $A \setminus \mathfrak{p}_i$. Thus, for all i there exists $x_i \in \operatorname{Ann}(f)$ such that $x_i \notin \mathfrak{p}_i$. It follows by an easy exercise that there exists $x \in \operatorname{Ann}(f)$ such that $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. But then x is not a zero divisor, by ([**L 2**], X Prop. 2.9). This can happen only if $f = 0$, so $\mathfrak{a} = 0$ and thus U is schematically dense in X.

Conversely, suppose that U does not contain the point corresponding to $\mathfrak{p}_i$. After renumbering the indices, we may assume that $i = n$ and that $\{i \mid \mathfrak{p}_i \not\supseteq \mathfrak{p}_n\} = \{1, \ldots, r\}$ for some $r < n$. Let

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r.$$

Since the chosen primary decomposition was minimal, we have $\mathfrak{a} \neq (0)$. We claim that $U \subseteq \operatorname{Spec} A/\mathfrak{a}$. Indeed, let $\mathfrak{p}$ be a prime ideal corresponding to a point in U. Since U is open and since $\mathfrak{p}_n \notin U$, we have $\mathfrak{p} \not\supseteq \mathfrak{p}_n$, and therefore $\mathfrak{p} \not\supseteq \mathfrak{p}_i$ for all $i > r$. Let $S = A \setminus \mathfrak{p}$. By ([**A-M**], Prop. 4.8(i)) and ([**A-M**], Prop. 4.9), we then have

$$(0) = \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i = \bigcap_{i=1}^{r} S^{-1}\mathfrak{q}_i = S^{-1}\mathfrak{a}.$$

Thus $\mathfrak{p} \in \operatorname{Spec} A/\mathfrak{a}$ (since $S^{-1}\mathfrak{a} = (0)$ implies $\mathfrak{a} \subseteq \mathfrak{p}$), and $(A/\mathfrak{a})_{\mathfrak{p}} = A_{\mathfrak{p}}$. This holds for all $\mathfrak{p} \in U$, so U is an open subscheme of $\operatorname{Spec} A/\mathfrak{a}$. This shows that U is not schematically dense in X. □

Proposition 3.3. *Let U be an open dense subscheme of a noetherian scheme X. Then X is reduced, irreducible, or integral if and only if U has the same property.*

Proof. The forward implication is trivial. If U is reduced, then $X_{\rm red}$ is a closed subscheme of X containing U, so $X = X_{\rm red}$ and therefore X is reduced. If U is irreducible, then so is X because the generic points of all irreducible components of X lie in U, by Proposition 3.2. □

Proposition 3.4. *A locally noetherian, reduced scheme has no embedded points.*

Proof. This is a well-known fact; see ([**E**], Exercise 11.10) for more details.

The question is local, so it is sufficient to consider the affine case, say $X = \operatorname{Spec} A$. Since A is noetherian, its nilradical is the intersection of the minimal primes ([**B**], Ch. II § 2, Prop. 13). Since A is reduced, this gives a primary decomposition for the zero ideal. □

As a consequence, for a locally noetherian scheme X, passing to $X_{\rm red}$ eliminates all the embedded points but leaves the underlying topological space unchanged.

Proposition 3.5. *Let X and Y be schemes over a scheme S, with Y separated, and let $f, g\colon X \to Y$ be morphisms over S. If f and g agree on a schematically dense open subset of X, then $f = g$.*

Proof. Let U be such an open subset. We have a morphism $(f, g)\colon X \to Y \times_S Y$; the inverse image of the (closed) diagonal subscheme of $Y \times_S Y$ is a closed subscheme of X containing U; hence it is all of X and therefore (f, g) factors through the diagonal. This implies that $f = g$. □

Definition 3.6. Let X be a scheme.

(a). If X is noetherian, then a **rational map** $f\colon X \dashrightarrow Y$ from X to another scheme Y is an equivalence class of pairs (U, f), where U is a schematically dense open subset of X, $f\colon U \to Y$ is a morphism, and (U, f) is said to be equivalent to (U', f') if f and f' agree on $U \cap U'$.

(b). If X is noetherian, then a **rational function** on X is a class of pairs (U, f), where U is a schematically dense open subset of X, f is a regular function on U, and the equivalence relation is similar to the one in part (a).

(c). A **rational section** of a sheaf $\mathscr{F}$ on X is a section of $\mathscr{F}$ over a schematically dense open set.

(d). A rational section of a line sheaf $\mathscr{L}$ on X is said to be **invertible** if it has an inverse over a schematically dense open subset.

Remark. The relations in parts (a) and (b) above are equivalence relations because Y is separated, and because the open sets in question are schematically dense.

When working with rational maps, the following results are often useful.

Proposition 3.7. *Let S be a locally noetherian scheme, let X be a normal scheme, separated and of finite type over S, let Y be a proper scheme over S, and let $f\colon X \dashrightarrow Y$ be a rational map over S. Then the set of points in X where f is not defined has codimension ≥ 2.*

Proof. See ([**EGA**], II 7.3.6 and 7.3.7). □

Corollary 3.8. *Let k be a field, let X be a normal curve over k, and let Y be a proper scheme over k. Then any rational map $X \dashrightarrow Y$ over k extends to a morphism.*

Proof. Immediate. □

The following proposition makes rigorous the intuition that, in most common cases, the points outside of a given open affine are "at infinity."

Proposition 3.9. *Let X be an integral scheme, separated and of finite type over an affine noetherian scheme $Y = \operatorname{Spec} B$. Let $U = \operatorname{Spec} A$ be an open affine subset of X. Then, for any point $P \in X$ with $P \notin U$, there is some function $f \in A$ which is not regular at P. Moreover, f can be chosen from among any given generating set of A over B.*

Proof. Let $\{x_1, \ldots, x_n\}$ be a generating set for A over B, and suppose that all x_i are regular at P. Then there is an open affine neighborhood $U' = \operatorname{Spec} A'$ of P such that all x_i are regular on U'. Hence $x_1, \ldots, x_n$ all lie in A', so $A \subseteq A'$. This gives a morphism $\phi\colon U' \to U$ which coincides with the identity map on $U \cap U'$. Since X is separated, ϕ equals the identity on the closure of the dense set $U \cap U'$, hence ϕ is an inclusion map $U' \hookrightarrow U$. In particular $U' \subseteq U$ and hence $P \in U$, a contradiction. Thus if $P \notin U$ then some x_i is not regular at P. □

In general, the generic fiber of a morphism is not a subscheme. However, it is often dense in the scheme, in a way that is made precise below. We now provide counterparts to Propositions 3.2, 3.3, and 3.5, in this situation.

Definition 3.10. A morphism is **scheme-theoretically dominant** if its schematic image is equal to its codomain.

A morphism $f\colon X \to Y$ is scheme-theoretically dominant if and only if the sheaf $\ker(\mathscr{O}_Y \to f_*\mathscr{O}_X)$ contains no nontrivial quasi-coherent subsheaves. See [**Stacks 01R6**].

Proposition 3.11. *Let Y be a regular integral noetherian scheme of dimension 1, let $k = K(Y)$, let $\pi\colon X \to Y$ be a separated morphism of finite type, and let X_k denote the generic fiber of π. Then the following conditions are equivalent.*

(i). *The map $X_k \to X$ is scheme-theoretically dominant;*
(ii). *π is flat;*
(iii). *X has no associated points outside of the generic fiber of π.*

Now assume that the above conditions hold. Then X is reduced, irreducible, or integral if and only if X_k has the same property.

Proof. We first show that (iii) $\implies$ (ii) $\implies$ (i) $\implies$ (iii). The first implication follows from ([**H 2**], III Prop. 9.7).

Assume that X is flat over Y. Since flatness and schematic images are both local, we may assume that X is affine, say $X = \operatorname{Spec} A$, and that X lies over an open affine

Spec B of Y. The field of fractions of B is again k. Showing that X is the schematic image of $X_k \to X$ is equivalent to showing that $A \to A \otimes_B k$ is injective. But this follows by applying the flatness of A over B to the injection $B \hookrightarrow k$. Thus (i) holds.

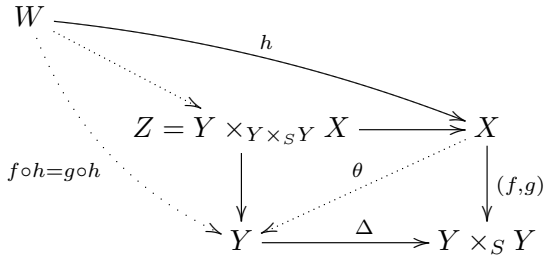
Next assume that (i) holds, and that X has an associated point ξ outside of the generic fiber. Let $Y' = Y \setminus \{\pi(\xi)\}$, and let $X' = X \times_Y Y'$. Then X' is an open subscheme of X. Let X'' be the schematic closure of X' in X. Then X'' is a closed subscheme of X with the same generic fiber, so by (i), $X'' = X$. But, by Proposition 3.2, $\xi \notin X''$, a contradiction. Thus (iii) holds.

Finally, assume that (i)–(iii) hold. If X_k is irreducible, then X has only one associated point by (iii), so it too is irreducible. The converse is clear. If X_k is reduced, then $X_k \to X$ factors through the closed subscheme X_{red} of X, so by (i) X is reduced. Again, the converse holds trivially. Therefore the last assertion holds. □

Proposition 3.5 can be generalized as follows.

Proposition 3.12 ([**EGA**], I 9.5.6). *Let X and Y be schemes over a scheme S, with Y separated over S. Let $f, g\colon X \to Y$ be morphisms over S. If there exist a scheme W and a scheme-theoretically dominant morphism $h\colon W \to X$ such that $f \circ h = g \circ h$, then $f = g$.*

Proof. Since $Y \to S$ is separated, the diagonal Δ is a closed subscheme of $Y \times_S Y$. Therefore its inverse image under the map $(f, g)\colon X \to Y \times_S Y$ (equal to the product $Y \times_{Y \times_S Y} X$; ([**EGA**], I 4.4.1)) is a closed subscheme Z of X. Since $f \circ h = g \circ h$, the map $(f, g) \circ h = (f \circ h, g \circ h)\colon W \to Y \times_S Y$ factors through Δ. Therefore, by the universal property of the product, h factors through Z.



This implies that Z contains the scheme-theoretic image of h, so $Z = X$. Thus, (f, g) factors through Δ; i.e., there is a map $\theta\colon X \to Y$ such that $(f, g) = \Delta \circ \theta$. Let $p_1, p_2\colon Y \times_S Y \to Y$ be the projection maps. Then $f = p_1 \circ (f, g) = p_1 \circ \Delta \circ \theta = \theta$ and similarly $g = \theta$; hence $f = g$. □

As corollaries, we have the following.

Corollary 3.13. *Let Y, $\pi\colon X \to Y$, and k be as in Proposition 3.11, and assume that X satisfies conditions (i)–(iii) of the proposition. Let X' be a separated scheme over Y. If $f, g\colon X \to X'$ are morphisms over Y that agree on X_k, then $f = g$.*

Proof. This is immediate from Proposition 3.12, since $X_k \to X$ is scheme-theoretically dominant. □

Corollary 3.14. *Let X and Y be varieties over a field k, and let $f, g\colon X \to Y$ be morphisms over k. If f and g agree on a Zariski-dense set of points, then $f = g$.*

Proof. Let $\Sigma \subseteq X$ be a Zariski-dense set of points such that $f(x) = g(x)$ for all $x \in \Sigma$. Let W be the scheme consisting of the disjoint union of $\operatorname{Spec} \boldsymbol{\kappa}(x)$ for all $x \in \Sigma$, and let $h\colon W \to X$ be the morphism taking $x \in \Sigma$ to $x \in X$, in the obvious way, for all $x \in \Sigma$. Since X is reduced and Σ is dense, h is scheme-theoretically dense. The assumptions imply that $f \circ h = g \circ h$, so $f = g$. □

§4. Cartier divisors and associated line sheaves

The above definition of invertible rational section of a line sheaf (Definition 3.6d) meshes well with the notion of a Cartier divisor.

Cartier divisors

We begin by recalling the definition of a Cartier divisor.

Definition 4.1. Let X be a scheme.

(a). The sheaf $\mathscr{K}$, called the **sheaf of total quotient rings** of $\mathscr{O}_X$, is the sheaf associated to the presheaf $U \mapsto S^{-1}\mathscr{O}_X(U)$, where S is the multiplicative system of nonzero elements which are not zero divisors. It contains $\mathscr{O}_X$ as a subsheaf of rings.

(b). Let $\mathscr{O}_X^*$ denote the sheaf of invertible elements of $\mathscr{O}_X$. Then a **Cartier divisor** on X is a global section of the sheaf $\mathscr{K}^*/\mathscr{O}_X^*$.

The set of Cartier divisors forms a group, which is written additively instead of multiplicatively by analogy with Weil divisors.

By standard properties of sheaves, a Cartier divisor can be described by giving an open cover $\{U_i\}$ of X and sections $f_i \in \mathscr{K}(U_i)$ such that f_i/f_j lies in $\mathscr{O}_X(U_i \cap U_j)^*$ for all i and j. Such a description is called a **system of representatives** for the Cartier divisor. If $\{(U_i, f_i)\}_{i\in I}$ is a system of representatives for a Cartier divisor D, then $D\big|_{U_i} = (f_i)$ for all i. If the scheme X is integral, which is true in most applications, then $\mathscr{K}$ is just the constant sheaf $K(X)$, and therefore in a system of representatives all f_i may be taken to lie in $K(X)$.

A Cartier divisor is **principal** if it lies in the image of the map

$$\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathscr{O}_X^*).$$

If X is integral, then this is equivalent to the assertion that all f_i can be taken equal to the same $f \in K(X)$. Two Cartier divisors are **linearly equivalent** if their difference is principal. A Cartier divisor is **effective** if it has a system of representatives $\{(U_i, f_i)\}_{i\in I}$ such that f_i lies in $\mathscr{O}_X(U_i)$ for all i. If D is a Cartier divisor, its **support**, denoted $\operatorname{Supp} D$, is the set

$$\operatorname{Supp} D = \{P \in X \mid D_P \notin \mathscr{O}_{X,P}^*\}$$

(where $D_P \in \mathscr{K}_P$ denotes the germ of D at P). Note that the group of units $\mathscr{O}_{X,P}^*$ in the local ring $\mathscr{O}_{X,P}$ coincides with the stalk $(\mathscr{O}_X^*)_P$ of the sheaf $\mathscr{O}_X^* \subseteq \mathscr{K}$. The support of D is a Zariski-closed subset of X. More concretely, if $\{(U_i, f_i)\}_{i\in I}$ is a system of representatives for D, then $(\operatorname{Supp} D) \cap U_i$ equals the set of all $P \in U_i$ such that f_i is not an invertible element of the local ring $\mathscr{O}_{X,P}$ at P. If D is a Cartier divisor with $\operatorname{Supp} D = \emptyset$, then $D = (1)$.

For more information on Cartier divisors and how they compare to Weil divisors, see ([**H 2**], II §6).

Pull-backs of Cartier divisors

A Cartier divisor cannot always be pulled back via a morphism; for example, if the support of the divisor contains the image of the morphism, then no such pull-back exists. A more precise criterion of when a pull-back exists can be found by looking at associated points.

Lemma 4.2. *Let A be a commutative ring, let $\mathfrak{a}$ be an ideal of A, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be prime ideals of A with $\mathfrak{p}_i \not\supseteq \mathfrak{a}$ for all i. Then there exists $f \in \mathfrak{a}$ not lying in any of the $\mathfrak{p}_i$.*

Proof. Exercise. See also ([**B**], Ch. II, § 1, Prop. 2). □

Lemma 4.3. *Let A be a commutative noetherian ring, let S be the multiplicative system of nonzero elements of A which are not zero divisors, and let $X = \operatorname{Spec} A$.*

(a). *An element $f \in A$ is a zero divisor or zero if and only if it is contained in some associated prime of A. Thus an element $f \in A$ lies in S if and only if the principal open subset $D(f)$ contains all associated points of X.*

(b). *Let U be an open subset of X containing all associated points. Then U can be written as a union of principal open subsets $D(f)$ with $f \in S$.*

(c). *If $U_1 \subseteq U_2$ are open subsets as in part (b), then the map $\mathscr{O}_X(U_2) \to \mathscr{O}_X(U_1)$ is injective.*

(d). *Let U be an open subset as in part (b). Then there is a natural injection $\mathscr{O}_X(U) \to S^{-1}A$.*

(e). *Conversely, for any $\sigma \in S^{-1}A$, there is an open subset U of X such that σ lies in $\mathscr{O}_X(U)$ (via the map of part (d)).*

(f). *We have $(S^{-1}A)^* \subseteq A_{\mathfrak{p}}^*$ for all associated primes $\mathfrak{p}$ of A.*

Proof. Part (a) is ([**L 2**], X Prop. 2.9).

Let $\mathfrak{a}$ be an ideal in A corresponding to the closed subset $X \setminus U$, let $\mathfrak{p}_0$ be any prime in U, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the associated primes of A. Then part (b) follows by applying Lemma 4.2 to $\mathfrak{a}$ and to $\mathfrak{p}_0, \dots, \mathfrak{p}_r$.

For part (c), assume the map is not injective and pick a nonzero section in $\mathscr{O}_X(U_2)$ restricting to zero in $\mathscr{O}_X(U_1)$. By part (b), we may assume that $U_2 = D(f_2)$ for some $f_2 \in S$, and that likewise $U_1 = D(f_1)$. But, since f_1 is not a zero divisor, the map $A_{f_2} \to A_{f_1}$ is injective, a contradiction.

Part (d) is easy to see, since we may find $D(f) \subseteq U$ with $f \in S$, and then compose the injections $\mathscr{O}_X(U) \to A_f \to S^{-1}A$.

For part (e), write $\sigma = a/s$ with $a \in A$ and $s \in S$; then we may take $U = D(s)$.

Finally, for part (f), let s_1/s_2 be an element of $(S^{-1}A)^*$. Then $s_1, s_2 \in S$, so by part (a), $s_1, s_2 \in A \setminus \mathfrak{p}$. Thus $s_1/s_2 \in A^*_{\mathfrak{p}}$. □

Proposition 4.4. *Let $\phi\colon X \to Y$ be a morphism of noetherian schemes, and let D be a Cartier divisor on Y. If the support of D does not contain the image of any associated point of X, then D lifts to a Cartier divisor ϕ^*D on X.*

Proof. We may assume that X and Y are affine, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Let S (resp. T) be the multiplicative system of nonzero elements of A (resp. B) which are not zero divisors. We may assume that D is given on Y by an element $f \in S^{-1}A$. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the associated primes of B, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be their respective inverse images in A, and let $\mathfrak{p}_{r+1}, \dots, \mathfrak{p}_{r'}$ be the associated primes of A. Let $\mathfrak{a}$ be the ideal in A given by

$$\mathfrak{a} = \{a \in A : af \in A\} .$$

Since the support of D does not contain any of the $\mathfrak{p}_i$ (by assumption and by Lemma 4.3f), we have $f \in A^*_{\mathfrak{p}_i}$ for all i, so $\mathfrak{p}_i \not\supseteq \mathfrak{a}$ for all i. By Lemma 4.2 there is an element $s \in \mathfrak{a}$ not lying in any of the $\mathfrak{p}_i$. Then $sf \in A$; also $s \in S$ and $\phi^*(s) \in T$ by Lemma 4.3a. Therefore f pulls back to the element $\phi^*(sf)/\phi^*(s) \in T^{-1}B$. Moreover, since $f \in A^*_{\mathfrak{p}_i}$ for all i, we have $sf \in A^*_{\mathfrak{p}_i}$ for all i, so in particular $sf \notin \mathfrak{p}_i$ for all i, and therefore $\phi^*(sf) \in T$. Thus f pulls back to an element of $\mathscr{K}^*$. □

The converse to the above proposition is false. For example, let $\phi\colon X \to Y$ be the morphism of schemes corresponding to the ring homomorphism

$$k[x, y] \to k[t, u]/(u^2, tu)$$

given by $x \mapsto t$, $y \mapsto t$, and let D be the Cartier divisor on $\operatorname{Spec} k[x, y]$ corresponding to x/y. Then the support of D contains the origin, which is the image of the embedded point in X, yet D lifts to the trivial divisor on X.

See also ([**EGA**], IV 21.4).

Cartier divisors and line sheaves

Definition 4.5. Let X be a noetherian scheme, let $\mathscr{L}$ be a line sheaf on X, and let s be an invertible rational section of $\mathscr{L}$. Then, for any open affine subset $U = \operatorname{Spec} A$ of X over which $\mathscr{L}$ is trivial, the restriction of s to U defines an element of $S^{-1}A$, where S is as in Lemma 4.3. Since s is invertible, this element actually lies in $(S^{-1}A)^*$. This element depends on the choice of trivialization: changing the trivialization multiplies the element by an element of A^*. Therefore, this gives a well-defined section of $\mathscr{K}^*/\mathscr{O}^*$ over U. These sections glue to give a global section of $\mathscr{K}^*/\mathscr{O}^*$ over X. This section, regarded as a Cartier divisor, is called the **(Cartier) divisor of** s, and is denoted (s).

Proposition 4.6. *Let X be a noetherian scheme.*

(a). *Let f be a rational function on X. Then f may be regarded as an invertible rational section of the trivial line sheaf $\mathscr{O}_X$, and the definition of (f) as such coincides with the principal divisor (f).*

(b). *Let* $\mathscr{L}$ *be a line sheaf on* X *and let* s *be an invertible rational section of* $\mathscr{L}$. *Then* s *is a global (regular) section of* $\mathscr{L}$ *if and only if* (s) *is effective.*

(c). *Let* $\phi\colon X' \to X$ *be a morphism of noetherian schemes, let* $\mathscr{L}$ *be a line sheaf on* X, *and let* s *be an invertible rational section of* $\mathscr{L}$ *which is defined and nonzero at the images (under* ϕ*) of all associated points of* X'. *Then* ϕ^*s *is an invertible rational section of* $\phi^*\mathscr{L}$, *and* $(\phi^*s) = \phi^*(s)$.

(d). *Let* s_1 *and* s_2 *be invertible rational sections of line sheaves* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *respectively. Then* $s_1 \otimes s_2$ *is an invertible rational section of* $\mathscr{L}_1 \otimes \mathscr{L}_2$, *and* $(s_1 \otimes s_2) = (s_1) + (s_2)$.

Proof. These all follow immediately from the definition. □

Recall from ([**H 2**], II §6) the definition of the sheaf $\mathscr{O}(D)$ associated to a Cartier divisor D on a scheme X (Hartshorne denotes it $\mathscr{L}(D)$). It is a subsheaf of $\mathscr{K}$.

Definition 4.7. Let D be a Cartier divisor on a noetherian scheme X. The global section $1 \in \mathscr{K}$ defines a rational section, called the **canonical section** of $\mathscr{O}(D)$, and denoted 1_D.

In addition to ([**H 2**], II Prop. 6.13), we have the following properties.

Proposition 4.8. *Let* D *be a Cartier divisor on a noetherian scheme* X.

(a). *If* s *is the canonical section of* $\mathscr{O}(D)$, *then* $(s) = D$.

(b). *If* $\phi\colon X' \to X$ *is a morphism of noetherian schemes such that none of the associated points of* X' *are taken into the support of* D, *then* $\mathscr{O}(\phi^*D) \cong \phi^*\mathscr{O}(D)$.

Proof. This is left as an exercise for the reader. □

Degrees of Cartier divisors on curves

Let X be a nonsingular curve over a field k, and let D be a Cartier divisor on X. Since X is locally factorial, we may regard D as equivalent to a Weil divisor: $D = \sum n_P \cdot P$, where the sum is taken over all closed points P of X, $n_P \in \mathbb{Z}$, and $n_P = 0$ for almost all P. Since k is not algebraically closed, the degree $\deg D$ is a little different from the definition in [**H**]:

Definition 4.9. With notation as above, the **degree** of D is the integer

$$\deg_k D := \sum_P n_P \cdot [k(P) : k] .$$

If the choice of k is clear from the context, then the subscript k may be omitted. For closed points $P \in X$, we also define the **local degree**:

$$\deg_{k,P} D = n_P \cdot [k(P) : k] ,$$

so that $\deg D = \sum \deg_P D$.

Proposition 4.10. *Let k be a field, let X and Y be nonsingular curves over k, let D be a divisor on Y, and let $f\colon X \to Y$ be a finite morphism. Recall that the degree of f is the degree $[K(X) : K(Y)]$ of the corresponding extension of function fields. Then:*

(a). *For all closed points $Q \in Y$,*

$$\sum_{P \in f^{-1}(Q)} \deg_P f^*D = (\deg f)(\deg_Q D)\,. \tag{4.10.1}$$

(b). $\deg f^*D = (\deg f)(\deg D)$

(c). *If Y is a complete curve and $f \in K(Y)^*$, then the degree of the principal divisor (f) is 0.*

Proof. By linearity, it suffices to prove part (a) when D is the single point Q with multiplicity 1.

By ([**EGA**], II 7.4.11), X and Y may be embedded as open subschemes of nonsingular projective curves X' and Y', respectively, and (by Corollary 3.8) f extends to a morphism $f'\colon X' \to Y'$.

Let $\operatorname{Spec} A$ be an open affine neighborhood of Q in Y; then $f^{-1}(\operatorname{Spec} A)$ is also affine, say $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$; moreover B is finite over A. We claim that all points in $(f')^{-1}(Q)$ lie in X. Indeed, suppose a point $P \in (f')^{-1}(Q)$ lies outside of X. Then it also lies outside of $\operatorname{Spec} B$, so by Proposition 3.9 there exists a function $f \in B$ which is not regular at P. Let

$$f^n + a_{n-1}f^{n-1} + \cdots + a_0 = 0$$

be an integral equation for f over A. Relative to the valuation corresponding to P, the first term of the above integral equation has (negative) valuation strictly smaller than the valuation of any of the other terms, leading to a contradiction. Thus the sum in (4.10.1) may be taken over all points of X' lying over Q.

By ([**EGA**], II 7.4.18), the closed points of a complete nonsingular curve over k are in bijection with the valuation rings of the function field containing k; hence in (4.10.1) the sum may be taken over all valuations of $K(X)$ extending the valuation of $K(Y)$ corresponding to Q. In this form, the equation follows from ([**B**], Ch. VI, § 8, Thm. 2).

See also ([**H 2**], II Prop. 6.9).

Part (b) follows immediately from (a) by summing over all Q.

For part (c), if $f \in k$ then $(f) = 0$ and the result is trivial. Otherwise, f determines a finite morphism $f\colon Y \to \mathbb{P}^1_k$, and D is the pull-back of the principal divisor (z) on $\mathbb{P}^1_k$. Thus, part (c) follows by part (b) and the trivial computation $\deg(z) = 0$. □

§5. The field of definition of a closed subscheme

In Weil-style algebraic geometry, a variety is an irreducible Zariski-closed subset of K^n or $\mathbb{P}^n(K)$ for some algebraically closed field K, and we say that the variety is defined over a *smaller* field k if its defining ideal can be generated by polynomials with coefficients in k. This convention is also often used in the study of elliptic curves.

In the present language, this definition can be phrased (and generalized) as follows. See also ([**EGA**], IV 4.8).

Definition 5.1. Let $k \subseteq k' \subseteq k_2$ be fields, let X be a scheme of finite type over k, and let Z be a closed subscheme of $X \times_k k_2$. Then we say that Z is **defined over** k' if there exists a subscheme Z' of $X \times_k k'$ such that $Z' \times_{k'} k_2 = Z$. If so, then we also say that k' is a **field of definition** for Z.

As was originally shown by Weil, in this situation there is a well-defined minimal field of definition.

Lemma 5.2. *Let $k \subseteq k_2$ be fields, let $n \in \mathbb{N}$, and let $\mathfrak{a}$ be an ideal in $k_2[X_1, \dots, X_n]$. Then there exists a field k_1 such that $k \subseteq k_1 \subseteq k_2$ and, for all fields k' with $k \subseteq k' \subseteq k_2$, $\mathfrak{a}$ is generated by elements of $k'[X_1, \dots, X_n]$ if and only if $k' \supseteq k_1$. Moreover, k_1 is finitely generated over k.*

Proof. By ([**C-L-O'S**], Ch. 2, § 7, Prop. 6), every ideal in a polynomial ring over a field has a unique reduced Gröbner basis. Let k_1 be the field generated over k by the coefficients of the elements of such a basis of $\mathfrak{a}$. Then the "if" part of the lemma is obvious, as well as the final assertion.

Conversely, suppose $\mathfrak{a}$ is generated by elements of $k'[X_1, \dots, X_n]$ for some field k'. Let $\mathfrak{a}'$ be the ideal in $k'[X_1, \dots, X_n]$ generated by those elements. Then $\mathfrak{a}'$ has a reduced Gröbner basis with coefficients in k'. But the definition of reduced Gröbner basis involves only linear algebra in the coefficients of the polynomials, so the unique reduced Gröbner basis is preserved by enlarging the field of coefficients. In particular the reduced Gröbner bases of $\mathfrak{a}'$ and $\mathfrak{a}$ coincide; hence $k' \supseteq k_1$. □

Proposition 5.3. *Let $k \subseteq k_2$ be fields, let X be a scheme of finite type over k, and let Z be a closed subscheme of $X \times_k k_2$. Then there exists a unique minimal field of definition k_1 of Z: for all fields k' with $k \subseteq k' \subseteq k_2$, Z is defined over k' if and only if $k' \supseteq k_1$. Moreover, k_1 is finitely generated over k.*

Proof. In the special case where $X = \mathbb{A}^n_k$, this follows immediately by translating Lemma 5.2 into geometrical language. The general case follows by covering X by finitely many open affines U_i, which can then be regarded as closed subschemes of $\mathbb{A}^{n_i}_k$. □

The following proposition gives a good idea of the structure of the field k_1.

Proposition 5.4. *Let k be a field, let $\mathfrak{p}$ be a prime ideal in $\bar{k}[X_1, \dots, X_n]$, and let k' be the smallest field such that $\mathfrak{p}$ is generated by elements of $k'[X_1, \dots, X_n]$. Let L be the field of fractions of $\bar{k}[X_1, \dots, X_n]/\mathfrak{p}$, and let K be the subfield of L*

generated by k and the images of $X_1, \ldots, X_n$. Then k' is a purely inseparable extension of the algebraic closure of k in K; moreover, K is separable over k if and only if k' is separable over k.

Proof. See ([**W**], Ch. 1, Prop. 23). □

Corollary 5.5. *Let X be a variety over a field k, and let k' be the minimal field of definition of X. Then k' is a purely inseparable extension of the algebraic closure of k in $K(X)$; moreover, $K(X)$ is separable over k if and only if k' is separable over k.*

Proof. This follows by translating Proposition 5.4 into geometrical language. □

§6. The field of definition of a variety

This section discusses what it means for a field to be a field of definition of a variety over a field, based on the concept of geometric integrality.

As noted in Remark 2.3, any variety over a number field k can also be given the structure of variety over $\mathbb{Q}$, so saying that it is defined over k would run counter to intuition. For example, let $X = \operatorname{Proj} \mathbb{Q}[x, y]/(y^2 - 2x^2)$. This is a closed subvariety of $\mathbb{P}^1_{\mathbb{Q}}$, so it is a variety over $\mathbb{Q}$, but intuition suggests that any "field of definition" of X should contain $\sqrt{2}$.

More abstractly, for any number field k, $\operatorname{Spec} k$ is a variety over $\mathbb{Q}$ (as well as over k). One expects that a "field of definition" of this variety would be k, and we show in this section that in fact it is, using the notion of geometric irreducibility. Note, however, that k need not be Galois over $\mathbb{Q}$; therefore the field of definition of an abstract variety need not be Galois over the base field.

We begin by describing what happens to a variety under base change to a larger field.

Throughout this section, if K/k is a field extension, and if X is a scheme over k, then X_K denotes $X \times_k K$. Likewise, if A is an algebra over k then A_K denotes the K-algebra $A \otimes_k K$.

Lemma 6.1. *Let $k \subseteq K$ be fields.*

(a). *If A is an algebra over k, then the natural map $A \to A_K$ is injective.*
(b). *If X is a scheme over k, then the natural projection $X_K \to X$ is surjective.*

Proof. Part (a) follows from the fact that A is flat over k.

For part (b), we have that X_K is faithfully flat over X ([**EGA**], IV 2.2.3 and IV 2.2.13(i)), and therefore $X_K \to X$ is surjective ([**EGA**], IV 2.2.6). □

The following two subsections explore issues related to the following definitions.

Definition 6.2. A scheme X over a field k is **geometrically irreducible** (resp. **geometrically reduced**, or **geometrically integral**) if $X_{\bar{k}}$ is irreducible (resp. reduced, or integral). All of these definitions are relative, in that they depend on k.

Separable field extensions and irreducible schemes

A base change by a separable field extension may cause an irreducible scheme to become reducible (and, for an arbitrary field extension, any additional irreducible components occur already after base change to a separable subextension).

Proposition 6.3. *Let X be a scheme of finite type over a field k, and let K be a purely inseparable field extension of k. Then the natural projection $\pi\colon X_K \to X$ induces a homeomorphism of the underlying topological spaces.*

Proof. We may assume that X is affine, say $X = \operatorname{Spec} A$. Let $p = \operatorname{char} k$; we may assume $p \ne 0$.

We first claim that, for all $f \in A_K$, there exists $e \in \mathbb{N}$ such that $f^{p^e} \in A$. Indeed, write f as a finite sum $f = \sum a_i \otimes x_i$ with $a_i \in A$ and $x_i \in K$ for all i. The x_i generate a finite extension of k, and so for some e we have $x_i^{p^e} \in k$ for all i. This value of e satisfies $f^{p^e} \in A$.

By Lemma 6.1b, $X_K \to X$ is surjective. Pick a prime $\mathfrak{p}$ of A and let $\mathfrak{q}$ be a prime of A_K lying over $\mathfrak{p}$. For $f \in A_K$, pick $e \in \mathbb{N}$ as in the claim; then $f \in \mathfrak{q}$ if and only if $f^{p^e} \in \mathfrak{q}$, which holds if and only if $f^{p^e} \in \mathfrak{p}$. Thus

$$\mathfrak{q} = \{f \in A_K : f^{p^e} \in \mathfrak{p} \text{ for some } e \in \mathbb{N}\}.$$

In particular, $X_K \to X$ is injective.

It remains only to show that $X_K \to X$ is a homeomorphism; i.e., the topology of X_K is not strictly finer than that of X. To show this, it suffices to consider only principal open subsets of X_K. Let $D(f)$ be such an open subset, and let e be as in the claim. Then $D(f) = D(f^{p^e})$, so it comes from an open subset in X since $f^{p^e} \in A$. □

Proposition 6.4. *Let X be a scheme of finite type over a field k. Then the following conditions are equivalent.*

(i). *$X_{\bar{k}}$ is irreducible.*
(ii). *X_{k^s} is irreducible, where k^s denotes the separable closure of k.*
(iii). *X_K is irreducible for all finite separable extensions K of k.*
(iv). *X_K is irreducible for all field extensions K of k.*

Proof. See ([**H 2**], II Ex. 3.15a) or ([**EGA**], IV 4.5.9). □

The following lemma gives some information on additional irreducible components that may arise from a base change.

Proposition 6.5. *Let X be an irreducible scheme of finite type over a field k, and let K be an extension field of k. Then every irreducible component of X_K dominates X.*

Proof. Since K is flat over k, the map $X_K \to X$ is also flat, so the going-down theorem ([**E**], Lemma 10.11) implies the lemma. □

Purely inseparable field extensions and reduced schemes

Parallel to the situation in the previous subsection, a base change by an inseparable field extension may cause a reduced scheme to lose that property.

Proposition 6.6. *Let X be a reduced scheme of finite type over a field k, and let K be a separable extension of k. Then X_K is also reduced.*

Proof. We start with a lemma.

Lemma 6.6.1. *Let A be an algebra over a field k, let K be a finite Galois extension of k, and let $\mathfrak{a}$ be an ideal in A_K. If $\mathfrak{a}$ is invariant under $\mathrm{Gal}(K/k)$, then there is an ideal $\mathfrak{a}_0$ of A such that $\mathfrak{a} = \mathfrak{a}_0 A_K$.*

Proof. Write $\mathrm{Gal}(K/k) = \{\sigma_1, \ldots, \sigma_n\}$. By the Normal Basis Theorem, there exists $x \in K$ such that $\sigma_1(x), \ldots, \sigma_n(x)$ is a basis for K over k. Let $a \in \mathfrak{a}$. We may write

$$a = \sum_{i=1}^{n} a_i \sigma_i(x) \ ,$$

with $a_i \in A$. It will suffice to show that the a_i actually lie in $\mathfrak{a}$.

We have

$$\begin{pmatrix} \sigma_1(a) \\ \vdots \\ \sigma_n(a) \end{pmatrix} = \begin{pmatrix} \sigma_1(\sigma_1(x)) & \ldots & \sigma_1(\sigma_n(x)) \\ \vdots & \ddots & \vdots \\ \sigma_n(\sigma_1(x)) & \ldots & \sigma_n(\sigma_n(x)) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \ .$$

By independence of characters, the matrix in the above equation is nonsingular. Hence the a_i can be written as linear combinations of the $\sigma_i(a)$ with coefficients in K. This shows that the a_i all lie in $\mathfrak{a}$, as was to be shown. □

Returning to the proof of the proposition, the property of being reduced is local, so it suffices to consider affine schemes $X = \mathrm{Spec}\, A$.

First consider the special case in which K is algebraic over k. Suppose that A_K contains a nonzero nilpotent element f. Then f lies in A_{K_0} for some finitely generated extension K_0 of k, so we may assume that K is finite over k. After replacing K with its normal closure over k, we may assume that K is finite Galois over k. Then $f \in A_K$, K is a finite Galois extension of k, and of course f is still nonzero and nilpotent in A_K.

Since the nilradical of A_K is invariant under $\mathrm{Gal}(K/k)$, Lemma 6.6.1 implies that it is generated by elements of A. In particular, X is not reduced, either, as was to be shown.

For the general case, it suffices to show that the proposition holds for a purely transcendental extension K/k, which is left as an exercise for the reader. □

Proposition 6.7. *Let X be a scheme of finite type over a field k. Then the following conditions are equivalent.*

(i). *$X_{\bar{k}}$ is reduced.*

(ii). *$X_{k^{\mathrm{perf}}}$ is reduced, where k^{perf} denotes the perfect closure of k.*
(iii). *X_K is reduced for all finite purely inseparable extensions K of k.*
(iv). *X_K is reduced for all field extensions K of k.*

Proof. See ([**H 2**], II Ex. 3.15b) or ([**EGA**], IV 4.6.1). □

Generalizing Proposition 6.5, we know that the associated points of a scheme do not go away after a base change to a larger field.

Proposition 6.8. *Let X be a scheme of finite type over a field k, and let K be a field extension of k. Then the image of the set of associated points of X_K under the projection $X_K \to X$ is the set of associated points of X.*

Proof. We may assume that X is affine, say $X = \operatorname{Spec} A$. Recall that $\operatorname{Ass}_A(M)$ denotes the set of associated primes of an A-module M, and that $\operatorname{Ass}_A(A)$ equals the set of associated primes of A. By ([**B**], Ch. IV, § 2, No. 6, Exemple 4),

$$\operatorname{Ass}_{A_K}(A_K) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A(A)} \operatorname{Ass}_{A_K}(A_K/\mathfrak{p}A_K) \,.$$

But by Lemma 6.1b the module $A_K/\mathfrak{p}A_K$ is nonzero for each $\mathfrak{p}$, so $\operatorname{Ass}_{A_K}(A_K/\mathfrak{p}A_K)$ contains at least one prime of A_K over $\mathfrak{p}$. □

Fields of definition

By making a base change to a larger field, a variety X may lose the property of being integral. This motivates the following definition.

Definition 6.9. Let X be a variety over a field k. We say that a field extension K of k is a **field of definition** for X if some irreducible component of X_K (with reduced induced subscheme structure) is geometrically integral.

Remark 6.10. In particular, k itself is a field of definition for X if and only if X is geometrically integral.

Lemma 6.11. *Let X be a variety over k, let K be a field extension of k, and let U be a nonempty open affine subvariety of X. Then K is a field of definition for X if and only if it is a field of definition for U.*

Proof. First assume that K is a field of definition for X. Then there exists an irreducible component X' of X_K which is geometrically integral. Letting $\pi\colon X_K \to X$ denote the canonical projection, Proposition 6.5 implies that $U' := \pi^{-1}(U) \cap X'$ is a nonempty open subset of X', which is therefore geometrically integral.

Conversely, if K is a field of definition for U, then $\pi^{-1}(U) = U_K$ has an irreducible component U' which is geometrically integral. Let X' be the closure of U' in X_K; then X' is an irreducible component of X_K.

To show that X' is geometrically integral, we first show that X' is geometrically irreducible. Let L be a field extension of K. Since X' and U' have the same generic point, the fibers of $X' \times_K L$ and $U' \times_K L$ over that point are the same, so

(by Proposition 6.5) they have the same number of irreducible components. Therefore $X' \times_K L$ is irreducible for all L, so X' is geometrically irreducible.

To show that X' is geometrically reduced, let L be a purely inseparable field extension of K. By Proposition 6.3, X'_L is irreducible. Let $\pi_L \colon X'_L \to X'$ be the projection map. Since U' is geometrically reduced, $U'_L = \pi_L^{-1}(U')$ is reduced. By Propositions 6.8 and 3.2, U'_L is schematically dense in X'_L. Therefore X'_L is reduced by Proposition 3.3. □

Regular field extensions

Definition 6.9 can be phrased in algebraic terms using the notion of regular field extensions.

Definition 6.12. A field extension K/k is **regular** if k is algebraically closed in K (*i.e.*, any element of K algebraic over k is already contained in k), and K is separable over k.

Lemma 6.13. *Let K/k be a field extension, and regard the algebraic closure $\bar{k}$ of k as a subfield of $\overline{K}$. Then the following conditions are equivalent.*

(i). *K is a regular extension of k;*
(ii). *K is linearly disjoint from $\bar{k}$ over k; and*
(iii). *the natural map $K \otimes_k \bar{k} \to K\bar{k}$ is injective.*

Proof. The equivalence (i) $\iff$ (ii) follows from ([**L 2**], Ch. VIII, Lemma 4.10). The equivalence of (ii) and (iii) is immediate from the definitions. □

Proposition 6.14. *Let X be a variety over k. Then k is a field of definition for X if and only if $K(X)$ is a regular extension of k.*

Proof. By Lemma 6.11, we may assume that X is affine.

Let A be the affine ring of X, and let $K = K(X)$. Then K is the field of fractions of A, and $X_{\bar{k}}$ is integral if and only if $A_{\bar{k}}$ is entire.

First suppose that K is regular over k. Consider the composition of maps

$$A \otimes_k \bar{k} \hookrightarrow K \otimes_k \bar{k} \to K\bar{k} \,,$$

where $\bar{k}$ is viewed as a subfield of an algebraic closure $\overline{K}$ of K. The first arrow is injective because $\bar{k}$ is flat over k. The second arrow is injective by Lemma 6.13, so $A \otimes_k \bar{k}$ is entire. Thus X is geometrically integral.

Conversely, assume that $A_{\bar{k}}$ is entire. By Lemma 6.13, it will suffice to show that K is linearly disjoint from $\bar{k}$ over k. Let $\theta_1, \ldots, \theta_n \in \bar{k}$, and assume that they are linearly independent over k. We need to show that they remain linearly independent over K.

Consider the composite map

$$A^n \to A_{\bar{k}} \to \overline{K} \,, \tag{6.14.1}$$

where the first map takes $(a_1, \dots, a_n)$ to $\sum a_i \otimes \theta_i$. This first map is injective by flatness of A over k and by injectivity of the map $k^n \to \bar{k}$ determined by the linearly independent elements $\theta_1, \dots, \theta_n$. By incomparability ([**E**], Cor. 4.18), since $A_{\bar{k}}$ is entire and integral over A, any map $A_{\bar{k}} \to \overline{K}$ extending the injection $A \hookrightarrow K$ remains injective. Therefore the composite map (6.14.1) is injective, and thus $\theta_1, \dots, \theta_n$ are linearly independent over K. □

Example 6.15 ([**Mac**], p. 384). This is an example of a field extension K/k which is not regular, even though k is algebraically closed in K.

Let F be a field of characteristic $p > 0$, let $k = F(x, y)$ with x and y indeterminates, and let K be the field of fractions of $k[u, v]/(xu^p + yv^p - 1)$.

We first claim that K is not separable over k. Indeed, $u, v, 1 \in K$ are linearly dependent over $k^{1/p}$ but not over k. Therefore K and $k^{1/p}$ are not linearly disjoint over k, so by MacLane's criterion K/k is not separable. (I thank Patrick Barrow for this shorter proof.[1])

We next claim that k is algebraically closed in K. Suppose not, and let α be an element of K that is algebraic over k but does not lie in k. Since $k(u)$ is purely transcendental over k, α is algebraic over $k(u)$, with the same degree and the same irreducible polynomial. Since K is purely inseparable over $k(u)$ of degree p and since $\alpha \in K$, α must also be purely inseparable over $k(u)$ of degree p, and so α is purely inseparable over k of degree p. Thus $K = k(\alpha, u)$, and therefore $K = k(\alpha)(u)$ is purely transcendental over $k(\alpha)$. In particular it is separable, so since $u, v, 1 \in K$ are linearly dependent over $k(\alpha)^{1/p}$, they must also be linearly dependent over $k(\alpha)$. The only linear dependence relation is $x^{1/p}u + y^{1/p}v = 1$, so some scalar multiple of this relation must have coefficients in $k(\alpha)$. Therefore $x^{1/p}$ and $y^{1/p}$ must lie in $k(\alpha)$, a contradiction since $[k(x^{1/p}, y^{1/p}) : k] = p^2 > p = [k(\alpha) : k]$. Thus k is algebraically closed in K, as was claimed.

The following criterion is often useful for showing that a scheme is geometrically integral. I thank G. Rémond ([**R**], p. 7) and B. Poonen for useful conversations about it.

Proposition 6.16. *Let X be an integral scheme of finite type over a field k. If X_{reg} contains a k-rational point, then X is geometrically integral.*

Proof. Let A be the local ring at some regular, k-rational point on X, and let $\mathfrak{m}$ be its maximal ideal; then $A/\mathfrak{m} = k$. We first show that k is algebraically closed in the fraction field $K(A)$. Indeed, let $\alpha \in K(A)$ be an algebraic element over k. Then it is integral over A, hence it lies in A since the local ring is normal. Since α is congruent mod $\mathfrak{m}$ to an element of k, we may subtract that element and assume that $\alpha \in \mathfrak{m}$. But now if $\alpha \neq 0$ then the same argument shows that α^{-1} lies in A, contradicting the fact that α cannot be a unit. Thus $\alpha \in k$, so k is algebraically closed in $K(A)$.

To show that the field extension $K(A)/k$ is separable, let d be the transcendence degree of $K(A)$ over k. By ([**E**], Ch. 8, Thm. A), $d = \dim A$, so by ([**E**], Ex. 16.15)

[1] Surprisingly, MacLane himself didn't use his own criterion.

$\Omega_{A/k}$ is free of rank d over A. Then $\Omega_{K(A)/k}$ is a vector space of dimension d over $K(A)$, so by ([**A**], Cor. 16.17a), $K(A)$ is separable over k. □

Remark 6.17. If k is perfect, then it suffices for X to have a normal rational point.

Example 6.18. Let $k = \mathbb{Q}$, and let $X = \operatorname{Spec} k[x,y]/(y^2 - 2x^2)$. Then X is integral but not geometrically integral, yet $X(k) \neq \emptyset$. Therefore Proposition 6.16 is false without any conditions on the rational point.

More stuff

——— This is not written yet. However, I may need the following (which is useful in its own right).

Proposition 6.19. *Let X be a variety over a field k, and let K be a normal field extension of k. Then all irreducible components of X_K are conjugate over k.*

Proof. By Proposition 6.5, we may assume that X is affine, say $X = \operatorname{Spec} A$. Write $A = k[x_1, \ldots, x_n]/P$ for some prime ideal P; then $X_K = \operatorname{Spec} A_K$, and

$$A_K \cong K[x_1, \ldots, x_n]/PK[x_1, \ldots, x_n]$$

since K is flat over k. Note that the ring $k[x_1, \ldots, x_n]$ is integrally closed and that $K[x_1, \ldots, x_n]$ is the integral closure of $k[x_1, \ldots, x_n]$ in $K(x_1, \ldots, x_n)$ (since it is integrally closed and integral over $k[x_1, \ldots, x_n]$). Therefore, by ([**Mat**], Thm. 9.3(iii)), all primes of $K[x_1, \ldots, x_n]$ lying over P are conjugate over $k(x_1, \ldots, x_n)$. By Proposition 6.5, these primes are exactly the primes corresponding to the irreducible components of X_K, so we conclude by noting that the restriction map

$$\operatorname{Gal}(K(x_1, \ldots, x_n)/k(x_1, \ldots, x_n)) \to \operatorname{Gal}(K/k)$$

is bijective. □

§7. Conventions and basic results in number theory

It is assumed that the reader has mastered the basics of algebraic number theory as presented, for example, in Part I of [**L 1**], especially the first five chapters.

In addition, the following definitions are used.

Number fields

A number field k has a canonical set of places, denoted M_k. This set is in one-to-one correspondence with the disjoint union of:

(i). the set of real embeddings $\sigma\colon k \hookrightarrow \mathbb{R}$;
(ii). the set of unordered complex conjugate pairs $\{\sigma, \bar{\sigma}\}$ of non-real embeddings $\sigma\colon k \hookrightarrow \mathbb{C}$; and
(iii). the set of nonzero prime ideals $\mathfrak{p}$ in the ring of integers R of k.

These places are referred to as **real places**, **complex places**, and **non-archimedean places**, respectively. In addition, an **archimedean place** is a real or complex place.

These places have **norms** $\|\cdot\|_v$ defined by

$$\|x\|_v = \begin{cases} |\sigma(x)|, & \text{if } v \text{ is real, corresponding to } \sigma\colon k \hookrightarrow \mathbb{R}; \\ |\sigma(x)|^2, & \text{if } v \text{ is complex, corresponding to } \sigma, \bar\sigma\colon k \hookrightarrow \mathbb{C}; \\ (R:\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)}, & \text{if } v \text{ is non-archimedean, corresponding to } \mathfrak{p} \subseteq R \end{cases}$$

(In the last of the above three cases, $\operatorname{ord}_{\mathfrak{p}}(x)$ denotes the **order** of x at $\mathfrak{p}$; *i.e.*, the exponent of $\mathfrak{p}$ in the factorization of the fractional ideal (x). This requires $x \ne 0$; we also define $\|0\|_v = 0$.) These are not necessarily genuine absolute values, since the norm associated to a complex place does not satisfy the triangle inequality.

Instead of the triangle inequality, however, we have that if $a_1, \dots, a_n \in k$, then

$$\left\| \sum_{i=1}^{n} a_i \right\|_v \le n^{N_v} \max_{1 \le i \le n} \|a_i\|_v, \tag{7.1}$$

where

$$N_v = \begin{cases} 1 & \text{if } v \text{ is real;} \\ 2 & \text{if } v \text{ is complex; and} \\ 0 & \text{if } v \text{ is non-archimedean.} \end{cases} \tag{7.2}$$

In particular, if v is non-archimedean, then $\|\cdot\|_v$ obeys something stronger than the triangle inequality:

$$\|x + y\|_v \le \max(\|x\|_v, \|y\|_v). \tag{7.3}$$

In addition, note that we have

$$\sum_{v \in M_k} N_v = [k : \mathbb{Q}] \tag{7.4}$$

for any number field k.

The set of archimedean and non-archimedean places of a number field k are denoted $M_k^\infty(k)$ and M_k^0, respectively.

Function fields

Definition 7.5. Let F be a field and let $n \in \mathbb{Z}_{>0}$. Then a **function field over F in n variables** is a finitely generated extension field of F of transcendence degree n.

Throughout this book, unless otherwise stated, function fields are assumed to be function fields in one variable.

For function fields in one variable, the following strengthens Prop. 2.4:

Proposition 7.6. *Let F be a field. Then the association $X \mapsto K(X)$ induces an arrow-reversing equivalence of categories between the category of nonsingular projective curves with dominant morphisms over F, and the category of function fields over F in one variable with maps over F.*

Proof. By ([**EGA**], II 7.4.18 and II 7.4.5), $X \mapsto K(X)$ maps onto the set of function fields over F (in one variable) (up to isomorphism). The sets of arrows in the two categories are equivalent by ([**EGA**], II 7.4.13).

Alternatively, the proof of ([**H 2**], I Cor. 6.12) extends readily to this case. □

Thus, a function field in one variable is the field $k := K(X)$ of rational functions on some nonsingular projective curve X over a field F.

Fix a constant $c > 1$. Then define a set M_k of places of k by defining, for each closed point $P \in X$, a place v with norm

$$\|x\|_v := c^{-[k(P):F]\,\mathrm{ord}_P\, x} \tag{7.7}$$

for all $x \neq 0$, where ord_P denotes the order of vanishing of the rational function x at P. (Again, we of course define $\|0\|_v = 0$ for all $v \in M_k$.) It is customary to take $c = e$, so that $\log \|x\|_v$ will take on integral values. Or, with finite fields, one could take $c = \#F$, providing some parallels with the number field case.

In the number field case, there is really only one choice for M_k, and the normalizations of $\|\cdot\|_v$ do not depend on additional choices. This is not the case in the function field case, however. Not only do the normalizations of $\|\cdot\|_v$ depend on c and on F, but also the set M_k may vary for a given function field k. For example, with the field $k = \mathbb{C}(X, Y)$, we may take $\mathbb{C}(X)$ as the field of constants and treat Y as an indeterminate, or vice versa. Therefore we will assume that, whenever a function field k is given, its set M_k of places is given, with norms at each of the places. Furthermore, an extension L/k of function fields is assumed to be one for which the set M_L is the set of all places extending places in M_k, and c and F coincide. Then, in the notation of the preceding paragraph, $L = K(X')$ for some curve X' provided with a finite morphism to X.

For a function field k, all places $v \in M_k$ are non-archimedean. Therefore, by (7.3), the set

$$\{x \in k \mid \|x\|_v \leq 1 \text{ for all } v \in M_k\}$$

is a subfield of k, called the **field of constants** of k. It is a finite extension of F (indeed, if k' is a maximal purely transcendental subextension of k/F, and k_1 is the algebraic closure of F in k, then $[k_1 : F] \leq [k : k']$). Therefore we often specify just the set of places and the corresponding norms when giving a function field in one variable.

For a function field k, let $N_v = 0$ for all $v \in M_k$; then (7.1) and (7.2) hold also in the function field case. In addition:

Convention 7.8. If k is a function field, then we adopt the convention that $[k : \mathbb{Q}] = 0$.

With this convention, (7.4) holds for function fields as well.

As with number fields, for a function field k we let M_k^∞ and M_k^0 denote the sets of archimedean and non-archimedean places in M_k, respectively. Since all places in M_k are non-archimedean when k is a function field, this means that $M_k^\infty = \emptyset$ and $M_k^0 = M_k$.

Global and local fields

Traditionally a global field is a number field or a function field in one variable over a finite field, and a local field is the completion of a global field at one of its places.

In this book, we are actually more interested in function fields of characteristic zero (mainly because Roth's theorem is false in positive characteristic), so we rarely use the terms global field or local field. Instead of "global field," we say number field or function field.

Likewise, instead of "local field," we say completion of a number field or function field at one of its places. Although a given function field k may have places not in M_k (e.g., $\mathbb{C}(X,Y)$), references to places of k will refer only to places in the chosen set M_k. If k is the completion of a number field or function field at a place v, then we let $M_k = \{v\}$.

The set of these (complete) fields is the disjoint union of:

(i). The set of finite extensions of $\mathbb{Q}_p$ for some rational prime p,
(ii). The set of finite extensions of $F((T))$ for some field F, and
(iii). $\mathbb{R}$ and $\mathbb{C}$.

If k is a number field or function field and $v \in M_k$, then the norm $\|\cdot\|_v$ extends uniquely to a norm on the completion k_v. In that case, we may drop the subscript v, since the place is implicit from the fact that we are dealing with elements of k_v: $\|x\|$ for $x \in k_v$.

If k is a number field, a function field, or the completion of such a field at one of its places, then we let $\overline{M}_k$ denote the disjoint union of M_L for all finite extensions L of k in a fixed algebraic closure.

We have the following basic results from commutative algebra.

Proposition 7.10. *Let A be a Dedekind ring, let k be its field of fractions, and let B be the integral closure of A in a finite extension L of k.*

$$\begin{array}{ccc} B & \subseteq & L \\ | & & | \\ A & \subseteq & k \end{array}$$

Then B is also a Dedekind ring. Also, if A is of finite type over a field, if $\operatorname{char} k = 0$, or if A is a complete local ring, then B is finite over A.

Proof. That B is Dedekind is a corollary of the Krull-Akizuki theorem ([**B**], Ch. VII, § 2, Cor. 2 of Prop. 5). Finiteness follows by ([**B**], Ch. IX, § 4, No. 1, Exemple), by ([**B**], Ch. IX, § 4, No. 1, Cor. of Prop. 1), and by ([**B**], Ch. IX, § 4, No. 2, Thm. 2). □

Note that the second assertion applies when k is a number field and A is its ring of integers, when k is a function field and A is the affine ring of some nonempty open affine in the associated projective curve, and when k is the completion of a number field or function field at one of its non-archimedean places and A is its valuation ring.

Note also that if L/k is inseparable, then *all* primes of A may ramify in B; an example is the extension $\mathbb{F}_p(T^{1/p})/\mathbb{F}_p(T)$.

Proposition 7.11. *Let k be a number field, a function field, or the completion of a number field or function field at one of its places. Let L be a finite extension of k, and let $v \in M_k$. Then*

$$\prod_{\substack{w \in M_L \\ w|v}} \|x\|_w = \|N_k^L x\|_v \qquad \text{for all } x \in L. \tag{7.11.1}$$

In particular,

$$\prod_{\substack{w \in M_L \\ w|v}} \|x\|_w = \|x\|_v^{[L:k]} \qquad \text{for all } x \in k. \tag{7.11.2}$$

Proof. The first equation (7.11.1) is well known in number theory in the case when L is separable over k. In general it is due to Chevalley ([**Ch**], Ch. 4, Thm. 1). It is true by ([**B**], Ch. VI, § 8, No. 5, eqn. (9) and Cor. 2) and ([**N**], II 4.8):

$$\|N_k^L x\|_v = \prod_{\substack{w \in M_L \\ w|v}} \left\|N_{k_v}^{L_w} x\right\|_v = \prod_{\substack{w \in M_L \\ w|v}} \left\|N_{k_v}^{L_w} x\right\|_w^{1/[L_w:k_v]} = \prod_{\substack{w \in M_L \\ w|v}} \|x\|_w .$$

The second equation follows easily from the first. □

Proposition 7.12 (Product Formula). *Let k be a number field or function field. Then*

$$\prod_{v \in M_k} \|x\|_v = 1 \qquad \text{for all } x \in k^*.$$

Proof. First we note that, for a finite extension L/k of global fields, the Product Formula for k implies the Product Formula for L. Indeed, for all $x \in L^*$,

$$\prod_{w \in M_L} \|x\|_w = \prod_{v \in M_k} \prod_{\substack{w \in M_L \\ w|v}} \|x\|_w = \prod_{v \in M_k} \left\|N_k^L x\right\|_v = 1$$

by (7.11.1). Thus it suffices to prove the Product Formula for $\mathbb{Q}$ and for $F(T)$. These are left to the reader as exercises. □

When k is a function field, Propositions 7.11 and 7.12 are equivalent to parts (a) and (c) of Proposition 4.10, respectively.

Artin and Whaples ([**A**], Ch. 12) showed that global fields (with a given collection of places and norms) are characterized by the product formula and certain "reasonableness" assumptions.

§8. Big divisors and Kodaira's lemma

Some of the tools of classification theory in algebraic geometry are very useful also in diophantine geometry. One example of this is the definition of "big divisors" and "big line sheaves."

We start with a lemma (which is useful to know in its own right).

Lemma 8.1. *Let $f\colon X' \to X$ be a finite morphism and let $\mathscr{L}$ be an ample line sheaf on X. Then $f^*\mathscr{L}$ is also ample.*

Proof. See ([**H 1**], Ch. I, Prop. 4.4). □

Theorem 8.2 ([**I Thm. 10.2**]). *Let X be a complete variety over a field k, and let $\mathscr{L}$ be a line sheaf on X. Assume that, for some integer $m > 0$, $\mathscr{L}^{\otimes m}$ has a nonzero global section, and let m_0 be the greatest common divisor of all such m. Then there is an integer $d \in \mathbb{N}$ and constants $c_2 \geq c_1 > 0$ such that*

$$c_1 n^d \leq h^0(X, \mathscr{L}^{\otimes nm_0}) \leq c_2 n^d \tag{8.2.1}$$

for all sufficiently large $n > 0$.

Proof. This follows Iitaka's proof quite closely, except that this proof allows X to be singular. By replacing $\mathscr{L}$ with $\mathscr{L}^{\otimes m_0}$, we may assume that $m_0 = 1$.

Let A be the set $\{m \in \mathbb{Z}_{>0} : h^0(X, \mathscr{L}^{\otimes m}) > 0\}$. Then A contains all sufficiently large integers. For each $m \in A$ let $\Phi_m\colon X \dashrightarrow \mathbb{P}^{r_m}_k$ be a rational map associated to some basis of $H^0(X, \mathscr{L}^{\otimes m})$ over k, and let W_m be the closure of its image. The function fields $K(W_m)$ are subfields of $K(X)$ containing k, and if $m, m' \in A$ then $K(W_{m+m'}) \supseteq K(W_m)$. Since $K(X)$ is a finitely generated field extension of k, it is a straightforward exercise to show that any ascending chain of subfields of $K(X)$ containing k must eventually stabilize; applying this to the set A ordered by $m_1 \geq m_2$ if $m_1 - m_2 \in A \cup \{0\}$ (which is a directed system), it follows that $K(W_m)$ is independent of m for all sufficiently large integers m. Fix one such integer m_1. Let d be the transcendence degree of $K(W_{m_1})$ over k.

We first show a lower bound for $h^0(X, \mathscr{L}^{\otimes nm_1})$, $n \in \mathbb{Z}_{>0}$. Let $\{s_0, \dots, s_r\}$ be the basis for $H^0(X, \mathscr{L}^{\otimes m_1})$ over k used to define Φ_{m_1}; then s_i/s_j is the element of $K(X)$ corresponding to the ratio x_i/x_j of coordinate functions on W_{m_1}, for all $i, j \in \{0, \dots, r\}$. Since the map $K(W_{m_1}) \to K(X)$ is injective, the maps $H^0(W_{m_1}, \mathscr{O}(n)) \to H^0(X, \mathscr{L}^{\otimes nm_1})$ are injective for all n. Since $\dim W_{m_1} = d$, we then have

$$h^0(X, \mathscr{L}^{\otimes nm_1}) \geq h^0(W_{m_1}, \mathscr{O}(n)) \geq c_1 n^d$$

for all $n \gg 0$, for some fixed $c_1 > 0$.

We next show a similar upper bound. Let X_1 be the closure of the graph of Φ_{m_1}, let $X_2 \to X_1$ be a morphism as in Chow's lemma, and let X' be the normalization of X_2. By Lemma 8.1, X' is a projective variety. We therefore have a birational

projective morphism $\alpha\colon X' \to X$, and a morphism $\phi_0\colon X' \to W_{m_1}$ such that the following diagram commutes.

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & W' \\ \downarrow{\scriptstyle\alpha} & \searrow^{\phi_0} & \downarrow{\scriptstyle\beta} \\ X & \overset{\Phi_{m_1}}{\dashrightarrow} & W_{m_1} \end{array}$$

Since X' is a projective variety, ϕ_0 is a projective morphism, so we can apply Stein factorization to get $\phi_0 = \beta \circ \phi$, with β finite and ϕ a morphism with connected fibers; moreover, $\phi_* \mathscr{O}_{X'} = \mathscr{O}_{W'}$ ([**H 2**], III Cor. 11.5 and proof). This latter fact also implies that the map $\Gamma(W', \mathscr{M}) \to \Gamma(X', \phi^* \mathscr{M})$ is an isomorphism for all line sheaves $\mathscr{M}$ on W'.

The remainder of this part of the proof will use the notation of divisors, often referring to Weil divisors. For Weil divisors D on X', let D_{hor} (resp. D_{ver}) denote the part of D composed of components that dominate W' via ϕ (resp. components that are contained in fibers of ϕ); then $D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$. Now fix a section $s_0 \in \Gamma(X, \mathscr{L}^{\otimes m_1})$, and let D be the Cartier divisor (s_0). We claim that, if $n \in \mathbb{Z}_{>0}$ and $s \in \Gamma(X, \mathscr{L}^{\otimes n m_1})$ is a global section, then $(\alpha^* s)_{\mathrm{hor}} = n(\alpha^* D)_{\mathrm{hor}}$. Indeed, taking horizontal part commutes with addition (and subtraction) of divisors, and the difference $(\alpha^* s) - \alpha^* D$ is the principal divisor generated by $s/s_0^n \in K(X) = K(X')$. This rational function lies in $K(W_{m_1})$, hence in $K(W')$, so the above principal divisor is the pull-back of a divisor on W'. Therefore it has no horizontal part. Thus the fixed part of the pull-back of the complete linear system $|nD|$ to X' contains $n(\alpha^* D)_{\mathrm{hor}}$, and therefore

$$\{f \in K(X) : (f) + nD \geq_{\mathrm{c}} 0\} \subseteq \{f \in K(X') : (f) + n(\alpha^* D)_{\mathrm{ver}} \geq_{\mathrm{w}} 0\}\,, \tag{8.2.2}$$

where the inequalities $\geq_{\mathrm{c}}$ and $\geq_{\mathrm{w}}$ refer to the cones of effective Cartier divisors and effective Weil divisors, respectively. (The inclusion may be strict due to effective Weil divisors on X' that do not come from effective Cartier divisors on X.)

Now, by Lemma 8.1, the pull-back of a hyperplane section of W_{m_1} to W' is ample; therefore by ampleness there is a Cartier divisor E on W' such that $\phi^* E \geq_{\mathrm{w}} (\alpha^* D)_{\mathrm{ver}}$. Hence

$$\begin{aligned} \{f \in K(X') : (f) + n(\alpha^* D)_{\mathrm{ver}} \geq_{\mathrm{w}} 0\} &\subseteq \{f \in K(X') : (f) + n\phi^* E \geq_{\mathrm{c}} 0\} \\ &= \{f \in K(W') : (f) + nE \geq_{\mathrm{c}} 0\}\,; \end{aligned} \tag{8.2.3}$$

note that since X' is normal, an effective Weil divisor that is also Cartier is effective as a Cartier divisor ([**H 2**], II Prop. 6.3A). Also note that the second step holds because $\phi_* \mathscr{O}_{X'} = \mathscr{O}_{W'}$. Combining (8.2.2) and (8.2.3), taking dimensions, and rewriting in terms of global sections of line sheaves gives

$$h^0(X, \mathscr{L}^{\otimes n m_1}) \leq h^0(W', \mathscr{O}(nE))\,.$$

But the right-hand side is bounded by a constant multiple of $n^{\dim W'} = n^d$. Indeed, we may add an effective divisor to E to make it very ample; then the bound follows by the theory of the Hilbert polynomial, or by Riemann-Roch.

Thus, (8.2.1) holds provided $m_1 \mid n$ (we are still assuming $m_0 = 1$). To prove the general case, first note that there is a fixed integer $a > 0$ such that for all n, there is some $q \in A$ such that $q \equiv n \pmod{m_1}$ and such that $am_1 - q$ also lies in A. Then, for $n \gg 0$,

$$\begin{aligned} c_1(n-q)^d &\le h^0(X, \mathscr{L}^{\otimes(n-q)}) \\ &\le h^0(X, \mathscr{L}^{\otimes n}) \\ &\le h^0(X, \mathscr{L}^{\otimes(n-q+am_1)}) \le c_2(n-q+am_1)^d\,, \end{aligned}$$

from which the general case of (8.2.1) follows readily. □

It is possible for $m_0 > 1$ to occur; for example, consider a line sheaf $\mathscr{L}$ corresponding to a nontrivial torsion point in $\operatorname{Pic}^0(X)$.

The above integer d always satisfies $d \le \dim X$. Indeed, pull back to a projective variety by Chow's lemma, and write $\mathscr{L}$ as $\mathscr{O}(D_1 - D_2)$ with D_1 ample and D_2 effective. Then

$$h^0(X, \mathscr{L}^{\otimes n}) \le h^0(X, \mathscr{O}(nD_1)) \ll n^{\dim X}\,.$$

In fact, the above bound holds for any projective scheme.

Definition 8.3. Let X, k, and $\mathscr{L}$ be as above. Then the **dimension** (or $\mathscr{L}$-dimension) $\kappa(\mathscr{L})$ of $\mathscr{L}$ is defined to be the integer $d \ge 0$ in Theorem 8.2 if some positive tensor power of $\mathscr{L}$ has a nonzero global section, and $-\infty$ otherwise. If D is a Cartier divisor on X, then the **dimension** (or D-dimension) $\kappa(D)$ of D is defined to be $\kappa(\mathscr{O}(D))$.

Thus, the Kodaira dimension of a nonsingular complete variety is just the dimension of its canonical line sheaf.

Definition 8.4. Let X and k be as above. Then a line sheaf $\mathscr{L}$ on X is **big** if $\kappa(L) = \dim X$. Likewise, a Cartier divisor D on X is **big** if $\mathscr{O}(D)$ is big.

Big divisors often behave like ample divisors outside of a proper Zariski-closed subset. For example, Lemma 8.1 has the following counterpart for big line sheaves and *generically* finite morphisms.

Proposition 8.5. *Let $f\colon X' \to X$ be a generically finite morphism and let $\mathscr{L}$ be a big line sheaf on X. Then $f^*\mathscr{L}$ is also big.*

Proof. Indeed, for all n the map

$$f^*\colon H^0(X, \mathscr{L}^{\otimes n}) \to H^0(X', f^*\mathscr{L}^{\otimes n})$$

is injective. □

On projective varieties, big divisors can be easily characterized in terms of ample and effective divisors.

Proposition 8.6 (Kodaira's lemma). *Let X be a projective variety over a field k, and let D be a Cartier divisor on X.*

(a). *If D is big, then for all divisors A on X, there is an integer $n > 0$ such that $nD - A$ is linearly equivalent to an effective divisor.*

(b). *The divisor D is big if and only if some positive multiple of it is linearly equivalent to a sum of an ample divisor and an effective divisor.*

Proof. First suppose that D is big. The assertion depends only on the linear equivalence class of A, so we may assume that $A = A_1 - A_2$, where A_1 and A_2 are effective Cartier divisors. Then it will suffice to show that $nD - A_1$ is linearly equivalent to an effective divisor.

Consider the exact sequence

$$0 \to H^0(X, \mathscr{O}(nD - A_1)) \to H^0(X, \mathscr{O}(nD)) \to H^0\big(A_1, \mathscr{O}(nD)\big|_{A_1}\big) .$$

The rank of the middle term grows like $n^{\dim X}$ for sufficiently large and divisible n, but the rank of the term on the right grows at most like $n^{\dim A_1} = n^{\dim X - 1}$. Therefore $H^0(X, \mathscr{O}(nD - A_1))$ must have positive rank for some sufficiently large and divisible n. This gives us (a).

The forward implication of part (b) follows immediately from part (a) (and in fact the ample divisor can be chosen to be any given ample divisor). The converse follows from the fact that ample divisors are big, and adding an effective divisor does not decrease the dimension of the divisor. □

§9. Descent

——— Write this. Should it go earlier? See Serre, Groupes Alg. et Corps de Classes, p. 108 (Ch. V No. 20).

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