1. Do Exercise #7 on page 15.

2. Let \( K = \mathbb{Q}(\sqrt[7]{7}, \sqrt[10]{10}) \). We will show that \( \mathcal{O}_K \) is not of the form \( \mathbb{Z}[\alpha] \) for any \( \alpha \in \mathcal{O}_K \).
   
   Fix \( \alpha \in \mathcal{O}_K \), let \( f(x) = \text{Irr}_{\alpha, \mathbb{Q}}(x) \), and let \( F_3 = \mathbb{Z}/3\mathbb{Z} \). For each \( g \in \mathbb{Z}[x] \) let \( \bar{g} \) be the polynomial in \( F_3[x] \) obtained by reducing the coefficients mod 3.
   
   (a). Show that \( g(\alpha) \) is divisible by 3 in \( \mathbb{Z}[\alpha] \) if and only if \( \bar{g} \) is divisible by \( \bar{f} \) in \( F_3[x] \).

   (b). Now suppose \( \mathcal{O}_K = \mathbb{Z}[\alpha] \). Consider the four algebraic integers

   \[
   \begin{align*}
   \alpha_1 &= (1 + \sqrt[7]{7})(1 + \sqrt[10]{10}) \\
   \alpha_2 &= (1 + \sqrt[7]{7})(1 - \sqrt[10]{10}) \\
   \alpha_3 &= (1 - \sqrt[7]{7})(1 + \sqrt[10]{10}) \\
   \alpha_4 &= (1 - \sqrt[7]{7})(1 - \sqrt[10]{10}).
   \end{align*}
   \]

   Show that all products \( \alpha_i \alpha_j \) (\( i \neq j \)) are divisible by 3 in \( \mathbb{Z}[\alpha] \), but that 3 does not divide any power of any \( \alpha_i \). (Hint: show that \( \alpha_i^n/3 \) is not an algebraic integer by considering its trace: show that

   \[
   \text{Tr}_{K/\mathbb{Q}}(\alpha_i^n) = \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n
   \]

   and that this is congruent mod 3 (in \( \mathbb{Z}[\alpha] \)) to

   \[
   (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^n = 4^n.
   \]

   Why does this imply that \( \text{Tr}_{K/\mathbb{Q}}(\alpha_i^n) \equiv 1 \pmod{3} \) in \( \mathbb{Z} \)?

   (c). Let \( \alpha_i = f_i(\alpha) \), \( f_i \in \mathbb{Z}[x] \) for each \( i = 1, 2, 3, 4 \). Show that \( \bar{f} \mid \bar{f}_i \bar{f}_j \) (\( i \neq j \)) in \( F_3[x] \) but \( \bar{f} \nmid \bar{f}_i^n \). Conclude that \( \bar{f} \) has an irreducible factor (over \( F_3 \)) which does not divide \( \bar{f}_i \) but which does divide all \( \bar{f}_j \), \( j \neq i \). (Recall that \( F_3[x] \) is a unique factorization domain.)

   (d). This shows that \( \bar{f} \) has at least four distinct irreducible factors over \( F_3 \). On the other hand \( \bar{f} \) has degree at most 4. Why is that a contradiction?