Math 254A. Homework 12

Due Wednesday, 15 November

1(NC). Find at least two roots of the polynomial $x^3 - 13x + 4$ in $\mathbb{Q}_2$. It will suffice to give inductive definitions of sequences converging to the given roots, and to show that the two roots are different. [**Hint:** use the method of the handout on Hensel's lemma.]

2. Describe the subgroup $(\mathbb{Q}_7^*)^2$ of $\mathbb{Q}_7^*$ explicitly in terms of p-adic series

$$a_m p^m + a_{m+1} p^{m+1} + \dots .$$

(Just describe the set, not the binary operation or inverse operation.)

3(NC). Show that the book's Hensel lemma (II (4.6)) implies the special case of the version of Hensel's lemma given in class on Friday, November 3.

(This special case says the following. Let $K = (K, |\cdot|)$ and A be as in the handout on Hensel's lemma, let $f \in A[x]$, and let $\alpha_0 \in A$. Assume that $|f(\alpha_0)| < 1$ and $|f'(\alpha_0)| = 1$. Then there exists $\alpha \in K$ such that $f(\alpha) = 0$ and $|\alpha - \alpha_0| < 1$.)

4. Show that there is a unique quadratic extension of $\mathbb{Q}_2$ whose valuation ring has residue field $\mathbb{F}_4$.

5. Let ξ be algebraic of degree n over $\mathbb{Q}_p$. Show that there is an integer N such that ξ does not satisfy any congruence

$$a_{n-1}\xi^{n-1} + a_{n-2}\xi^{n-2} + \cdots + a_0 \equiv 0 \pmod{p^N},$$

in which the a_i are rational integers, not all divisible by p. [**Hint:** Compactness.]