Synopsis of Class Field Theory
Roughly following Neukirch

Let $K$ be a number field, and denote places of $K$ by $p$. For finite places $p$, let $\mathcal{O}_p$ denote the valuation ring of $K_p$, and let

$$U_p = \begin{cases} \mathcal{O}_p^* & \text{if } p \nmid \infty, \\ \mathbb{R}_{>0}^* \subseteq \mathbb{R}^* = K_p & \text{if } p \text{ is real, and} \\ K_p^* & \text{if } p \text{ is complex.} \end{cases}$$

Then the group of idèles $I_K$ is the restricted direct product of $K_p^*$ with respect to the subgroups $U_p$. (This is not quite the book’s definition, due to differences at the real places, but there are only finitely many of those, so in the end it makes no difference.)

We have $K^* \subseteq I_K$ as a discrete subgroup (embedded diagonally).

For all finite sets $S$ of places of $K$ there are subgroups $I_{K}^{S} = \prod_{p \in S} K_p^* \times \prod_{p \not\in S} U_p$ of $I_K$. If $S \subseteq S'$ then $I_{K}^{S} \subseteq I_{K}^{S'}$. Also $I_{K} = \bigcup I_{K}^{S}$.

For $n \in \mathbb{N}$ we define subgroups of $K_p^*$:

$$V_p^{(n)} = \begin{cases} K_p^* & \text{if } p \text{ is finite and } n = 0, \\ 1 + p^n \mathcal{O}_p & \text{if } p \text{ is finite and } n > 0, \\ \mathbb{R}_{>0}^* & \text{if } p \text{ is real, and} \\ K_p^* & \text{if } p \text{ is complex.} \end{cases}$$

and

$$U_p^{(n)} = V_p^{(n)} \cap U_p = \begin{cases} \mathcal{O}_p^* & \text{if } p \text{ is finite and } n = 0, \\ 1 + p^n \mathcal{O}_p & \text{if } p \text{ is finite and } n > 0, \\ \mathbb{R}_{>0}^* & \text{if } p \text{ is real, and} \\ K_p^* & \text{if } p \text{ is complex.} \end{cases}$$

These groups are open subgroups of $U_p$ of finite index, and every open subgroup of $U_p$ contains $U_p^{(n)}$ for sufficiently large $n$.

A module $m$ of $K$ is a nonzero ideal in $\mathcal{O}_K$. We often write $m = \prod p^{n_p}$, and let $n_p = 0$ for infinite $p$. If $\alpha = (\alpha_p)_p$ and $\beta$ are idèles, then we say that $\alpha \equiv \beta \pmod{m}$ if $\alpha_p/\beta_p \in V_p^{(n_p)}$ for all $p$ (this is only a restriction for finite $p \mid m$ and for real places).

Let $J_{K}^m$ be the group of fractional ideals of $K$ prime to $m$, let $P_{K}^m$ be the group of principal fractional ideals of $K$ generated by some $x \in K$ with $x \equiv 1 \pmod{m}$, and let $\text{Cl}_{K}^m = J_{K}^m/P_{K}^m$. 

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The group of idèlle classes is \( C_K := I_K/K^* \).

There is a natural surjection \( I_K \twoheadrightarrow J_K \) with kernel \( I_K^{S\infty} \), so \( J_K \cong I_K/I_K^{S\infty} \) and \( \text{Cl}_K \cong I_K/K^* I_K^{S\infty} \cong C_K / (K^* I_K^{S\infty} / K^*) \).

In the Arakelov context, we have a natural surjection \( I_K \twoheadrightarrow \mathcal{J}(\mathcal{O}_K) \) with kernel
\[
I_K^0 = \prod \{ \alpha_p \in K_p^* : \|\alpha_p\| = 1 \}.
\]

For an idèlle \( \alpha \), we define its absolute norm \( N(\alpha) = \prod N(\mathfrak{p})^{v_p(\alpha)} = \prod \|\alpha_p\|^{-1} \).

Since \( N(x) = 1 \) for all \( x \in K^* \), this descends to a well-defined continuous function on \( C_K \), and we define \( C_K^0 = \{ \alpha \in C_K : N(\alpha) = 1 \} \). It is compact. Note that this is not a similar definition to \( I_{K_0}^1 \).

We let \( I_{K_0}^m = \prod U_{\mathfrak{p}}^{(n_{\mathfrak{p}})} \). Then \( C_{K_0}^m := K^* I_{K_0}^m / K^* \) is the congruence subgroup of \( C_K \), and \( C_K / C_{K_0}^m \) is the ray class group. We have \( C_K / C_{K_0}^1 \cong C_{K_0}^m \).

Let \( L/K \) be a finite extension of number fields. We define a norm \( N_{K}^L : I_L \to I_K \) by
\[
( N_{K}^L(\mathfrak{a}) )_{\mathfrak{p}} = \prod_{\mathfrak{q} | \mathfrak{p}} N_{K, \mathfrak{q}}^{L, \mathfrak{a}}.
\]

This map is continuous and has open image. It extends to a map \( C_L \to C_K \), also denoted \( N_{K}^L \).

Let \( L/K \) be an abelian extension and let \( G = \text{Gal}(L/K) \). For finite primes \( \mathfrak{p} \) of \( K \) not ramifying in \( L \), there is a well-defined Frobenius element \( \sigma \in G \) depending only on \( \mathfrak{p} \). We define the Artin symbol \( (\mathfrak{p}, L/K) \) to be this Frobenius element, and extend this multiplicatively to all fractional ideals of \( K \) not involving ramified primes.

A module \( m \) is admissible for \( L/K \) if \( N_{K_0}^L \mathfrak{m} \mathfrak{l} \supseteq U_{\mathfrak{p}}^{(n_{\mathfrak{p}})} \) for all places \( \mathfrak{p} \) of \( K \) and all (equivalently, at least one) \( \mathfrak{q} | \mathfrak{p} \). It is a deep theorem that there exists an admissible module involving only ramified primes.

If \( m \) is admissible for \( L/K \), then it is known that \( C_K / N_{K_0}^L C_L \cong J^m_{K_0} / P^m_{K_0} N_{K_0}^L J^m_{L_0} \), and that the Artin symbol is trivial on \( P^m_{K_0} \). Therefore, the Artin symbol defines an Artin map from \( C_K \) to \( \text{Gal}(L/K) \).

**Theorem** (Global Class Field Theory). Let \( K \) be a number field. Then:

(a) The map \( L \mapsto N_{K}^L C_L \) induces an inclusion-reversing one-to-one correspondence between finite abelian extensions \( L \) of \( K \) and open subgroups \( H \) of \( C_K \) (equivalently, closed subgroups of finite index).

(b) If \( L/K \) is finite abelian and \( H = N_{K}^L C_L \), then the Artin map induces an isomorphism \( C_K / H \cong \text{Gal}(L/K) \).

(c) If \( L \) and \( L' \) are finite abelian over \( K \) and \( H \) and \( H' \) are the corresponding subgroups of \( C_K \), then \( LL' \) corresponds to \( H \cap H' \) and \( L \cap L' \) corresponds to \( HH' \).

(d) A finite place \( \mathfrak{p} \) of \( K \) is unramified in \( L \) if and only if \( H \supseteq K^* U_{\mathfrak{p}} / K^* \) (where \( U_{\mathfrak{p}} \) is embedded in \( I_K \) by extending by 1’s at places other than \( \mathfrak{p} \)).

(e) A (finite or infinite) place \( \mathfrak{p} \) of \( K \) splits completely in \( L \) if and only if \( H \supseteq K^* K_{\mathfrak{p}}^\times / K^* \).