Synopsis of Class Field Theory

Roughly following Neukirch

Let K be a number field, and denote places of K by $\mathfrak{p}$.

For finite places $\mathfrak{p}$, let $\mathscr{O}_\mathfrak{p}$ denote the valuation ring of $K_\mathfrak{p}$, and let

$$U_\mathfrak{p} = \begin{cases} \mathscr{O}_\mathfrak{p}^* & \text{if } \mathfrak{p} \nmid \infty, \\ \mathbb{R}_{>0}^* \subseteq \mathbb{R}^* = K_\mathfrak{p} & \text{if } \mathfrak{p} \text{ is real, and} \\ K_\mathfrak{p}^* & \text{if } \mathfrak{p} \text{ is complex.} \end{cases}$$

Then the group of **idèles** I_K is the restricted direct product of $K_\mathfrak{p}^*$ with respect to the subgroups $U_\mathfrak{p}$. (This is not quite the book's definition, due to differences at the real places, but there are only finitely many of those, so in the end it makes no difference.)

We have $K^* \subseteq I_K$ as a discrete subgroup (embedded diagonally).

For all finite sets S of places of K there are subgroups

$$I_K^S = \prod_{\mathfrak{p} \in S} K_\mathfrak{p}^* \times \prod_{\mathfrak{p} \notin S} U_\mathfrak{p}$$

of I_K. If $S \subseteq S'$ then $I_K^S \subseteq I_K^{S'}$. Also $I_K = \bigcup I_K^S$.

For $n \in \mathbb{N}$ we define subgroups of $K_\mathfrak{p}^*$:

$$V_\mathfrak{p}^{(n)} = \begin{cases} K_\mathfrak{p}^* & \text{if } \mathfrak{p} \text{ is finite and } n = 0, \\ 1 + \mathfrak{p}^n \mathscr{O}_\mathfrak{p} & \text{if } \mathfrak{p} \text{ is finite and } n > 0, \\ \mathbb{R}_{>0}^* & \text{if } \mathfrak{p} \text{ is real, and} \\ K_\mathfrak{p}^* & \text{if } \mathfrak{p} \text{ is complex} \end{cases}$$

and

$$U_\mathfrak{p}^{(n)} = V_\mathfrak{p}^{(n)} \cap U_\mathfrak{p} = \begin{cases} \mathscr{O}_\mathfrak{p}^* & \text{if } \mathfrak{p} \text{ is finite and } n = 0, \\ 1 + \mathfrak{p}^n \mathscr{O}_\mathfrak{p} & \text{if } \mathfrak{p} \text{ is finite and } n > 0, \\ \mathbb{R}_{>0}^* & \text{if } \mathfrak{p} \text{ is real, and} \\ K_\mathfrak{p}^* & \text{if } \mathfrak{p} \text{ is complex.} \end{cases}$$

These groups are open subgroups of $U_\mathfrak{p}$ of finite index, and every open subgroup of $U_\mathfrak{p}$ contains $U_\mathfrak{p}^{(n)}$ for sufficiently large n.

A **module** $\mathfrak{m}$ of K is a nonzero ideal in $\mathscr{O}_K$. We often write $\mathfrak{m} = \prod \mathfrak{p}^{n_\mathfrak{p}}$, and let $n_\mathfrak{p} = 0$ for infinite $\mathfrak{p}$. If $\alpha = (\alpha_\mathfrak{p})_\mathfrak{p}$ and β are idèles, then we say that $\alpha \equiv \beta \pmod{^+ \mathfrak{m}}$ if $\alpha_\mathfrak{p}/\beta_\mathfrak{p} \in V_\mathfrak{p}^{(n_\mathfrak{p})}$ for all $\mathfrak{p}$ (this is only a restriction for finite $\mathfrak{p} \mid \mathfrak{m}$ and for real places).

Let $J_K^\mathfrak{m}$ be the group of fractional ideals of K prime to $\mathfrak{m}$, let $P_K^\mathfrak{m}$ be the group of principal fractional ideals of K generated by some $x \in K$ with $x \equiv 1 \pmod{^+ \mathfrak{m}}$, and let $\mathrm{Cl}_K^\mathfrak{m} = J_K^\mathfrak{m} / P_K^\mathfrak{m}$.

The group of **idèle classes** is $C_K := I_K/K^*$.

There is a natural surjection $I_K \twoheadrightarrow J_K$ with kernel $I_K^{S_\infty}$, so $J_K \cong I_K/I_K^{S_\infty}$ and $\mathrm{Cl}_K \cong I_K/K^* I_K^{S_\infty} \cong C_K \,/\, (K^* I_K^{S_\infty}/K^*)$.

In the Arakelov context, we have a natural surjection $I_K \twoheadrightarrow \overline{J}(\mathscr{O}_K)$ with kernel

$$I_K^0 = \prod \{\alpha_\mathfrak{p} \in K_\mathfrak{p}^* : \|\alpha_\mathfrak{p}\|_\mathfrak{p} = 1\} \;.$$

For an idèle α, we define its **absolute norm** $N(\alpha) = \prod N(\mathfrak{p})^{v_\mathfrak{p}(\alpha_\mathfrak{p})} = \prod \|\alpha_p\|_\mathfrak{p}^{-1}$. Since $N(x) = 1$ for all $x \in K^*$, this descends to a well-defined continuous function on C_K, and we define $C_K^0 = \{\alpha \in C_K : N(\alpha) = 1\}$. It is compact. Note that this is not a similar definition to I_K^0.

We let $I_K^\mathfrak{m} = \prod U_\mathfrak{p}^{(n_\mathfrak{p})}$. Then $C_K^\mathfrak{m} := K^* I_K^\mathfrak{m}/K^*$ is the **congruence subgroup** of C_K, and $C_K/C_K^\mathfrak{m}$ is the **ray class group**. We have $C_K/C_K^\mathfrak{m} \cong \mathrm{Cl}_K^\mathfrak{m}$.

Let L/K be a finite extension of number fields. We define a norm $N_K^L \colon I_L \to I_K$ by

$$\left(N_K^L \alpha\right)_\mathfrak{p} = \prod_{\mathfrak{q}|\mathfrak{p}} N_{K_\mathfrak{p}}^{L_\mathfrak{q}} \alpha_\mathfrak{q} \;.$$

This map is continuous and has open image. It extends to a map $C_L \to C_K$, also denoted N_K^L.

Let L/K be an abelian extension and let $G = \mathrm{Gal}(L/K)$. For finite primes $\mathfrak{p}$ of K not ramifying in L, there is a well-defined Frobenius element $\sigma \in G$ depending only on $\mathfrak{p}$. We define the **Artin symbol** $(\mathfrak{p}, L/K)$ to be this Frobenius element, and extend this multiplicatively to all fractional ideals of K not involving ramified primes. A module $\mathfrak{m}$ is **admissible** for L/K if $N_{K_\mathfrak{p}}^{L_\mathfrak{q}} L_\mathfrak{q}^* \supseteq U_\mathfrak{p}^{(n_\mathfrak{p})}$ for all places $\mathfrak{p}$ of K and all (equivalently, at least one) $\mathfrak{q} \mid \mathfrak{p}$. It is a deep theorem that there exists an admissible module involving only ramified primes.

If $\mathfrak{m}$ is admissible for L/K, then it is known that $C_K/N_K^L C_L \cong J_K^\mathfrak{m}/P_K^\mathfrak{m} N_K^L J_L^{\mathfrak{m}\mathscr{O}_L}$, and that the Artin symbol is trivial on $P_K^\mathfrak{m}$. Therefore, the Artin symbol defines an **Artin map** from C_K to $\mathrm{Gal}(L/K)$.

Theorem (Global Class Field Theory). *Let K be a number field. Then:*

(a). *The map $L \mapsto N_K^L C_L$ induces an inclusion-reversing one-to-one correspondence between finite abelian extensions L of K and open subgroups H of C_K (equivalently, closed subgroups of finite index).*

(b). *If L/K is finite abelian and $H = N_K^L C_L$, then the Artin map induces an isomorphism $C_K/H \xrightarrow{\sim} \mathrm{Gal}(L/K)$.*

(c). *If L and L' are finite abelian over K and H and H' are the corresponding subgroups of C_K, then LL' corresponds to $H \cap H'$ and $L \cap L'$ corresponds to HH'.*

(d). *A finite place $\mathfrak{p}$ of K is unramified in L if and only if $H \supseteq K^* U_\mathfrak{p}/K^*$ (where $U_\mathfrak{p}$ is embedded in I_K by extending by 1's at places other than $\mathfrak{p}$).*

(e). *A (finite or infinite) place $\mathfrak{p}$ of K splits completely in L if and only if $H \supseteq K^* K_\mathfrak{p}^*/K^*$.*