Math 254A. Absolute Values on Number Fields and Function Fields

This handout describes in more detail the situation with absolute values on number fields and function fields of dimension 1 over an arbitrary field.

Throughout this handout, all function fields are assumed to be of dimension 1 (i.e., transcendence degree 1) over a field of constants F.

We start by showing how the construction of the completion of a valued field behaves with respect to a field extension of valued fields.

Proposition 1. *Let L/K be a finite extension of valued fields such that the absolute value on L extends the absolute value on K. Then there is a commutative diagram*

$$\begin{array}{ccc} L & \xrightarrow{j} & \widehat{L} \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\psi} \\ K & \xrightarrow{i} & \widehat{K} \end{array}$$

in which ϕ is the inclusion map and i and j are the maps constructed in Theorem 5a in the handout "Valued rings and valued fields." Moreover, $\widehat{L} = j(L)\psi(\widehat{K})$.

Proof. Let i, j, and ϕ be as given. Then the existence of ψ such that the diagram commutes follows from Theorem 5b of the earlier handout. We regard all of the fields in the above diagram as valued subfields of $\widehat{L}$, so that all of the maps in the diagram are inclusion maps.

Now regard L as a vector space over K, and fix a basis. By II Prop. 4.9, a sequence in L is Cauchy (in the absolute value on L) if and only if all of the coordinate sequences with respect to this basis are Cauchy, and therefore this basis spans $\widehat{L}$ as a vector space over $\widehat{K}$. This proves the last assertion. $\square$

The following result was only partially proved in class (on 3 November).

Proposition 2. *Let K be a field. Then the map*

$$\sigma \in \operatorname{Hom}(K, \mathbb{C}) \mapsto (|x| = |\sigma(x)| \ \forall\, x \in K) \tag{2.1}$$

determines a well-defined bijection from $\operatorname{Hom}(K,\mathbb{C})/$(complex conjugation) to the set of archimedean places of K.

Proof. Clearly a map $\sigma\colon K \to \mathbb{C}$ and its complex conjugate determine the same absolute value on K, and this absolute value is archimedean, so the map (2.1) is well defined. It is surjective by a corollary of Ostrowski's theorem from class (November 3). It remains only to show that the map (2.1) is injective.

Let $\sigma_1, \sigma_2 \in \operatorname{Hom}(K,\mathbb{C})$ and assume that they are mapped to the same place of K. It was proved in class on November 1 that if two absolute values $|\cdot|_1$ and $|\cdot|_2$ on a field are equivalent, then $|\cdot|_1 = |\cdot|_2^s$ for some (fixed) $s > 0$. Therefore we have

$|\sigma_1(x)| = |\sigma_2(x)|^s$ for all $x \in K$. However, comparing the values at $x = 2$ gives that $s = 1$, so $|\sigma_1(x)| = |\sigma_2(x)|$ for all $x \in K$.

Now for any given $x \in K$, we have $|\sigma_1(x)| = |\sigma_2(x)|$ and $|\sigma_1(x-1)| = |\sigma_2(x-1)|$. Therefore, comparing the triangle with vertices $0, 1, \sigma_1(x) \in \mathbb{C}$ with the triangle with vertices $0, 1, \sigma_2(x) \in \mathbb{C}$, we find that both triangles have the same side lengths, so either $\sigma_1(x) = \sigma_2(x)$ or $\sigma_1(x) = \overline{\sigma_2(x)}$.

Now if $\sigma_1(K) \subseteq \mathbb{R}$, then by the above we also have $\sigma_2(K) \subseteq \mathbb{R}$, and therefore $\sigma_1 = \sigma_2$.

Hence we may assume that $\sigma_1(K) \nsubseteq \mathbb{R}$. Fix $x \in K$ such that $\sigma_1(x) \notin \mathbb{R}$. After replacing σ_2 with its complex conjugate if necessary, we may assume that $\sigma_2(x) = \sigma_1(x)$. Then we are done if $\sigma_1 = \sigma_2$. If not, then there is some $y \in K$ such that $\sigma_1(y) \ne \sigma_2(y)$. Then we must have $\sigma_1(y) = \overline{\sigma_2(y)}$. After replacing y with $1/y$ if necessary, we may assume that the imaginary parts of $\sigma_1(x)$ and $\sigma_1(y)$ have the same sign. Then the imaginary parts of $\sigma_2(x) = \sigma_1(x)$ and $\sigma_2(y) = \overline{\sigma_1(y)}$ have different signs, so

$$|\sigma_1(x-y)| = |\sigma_1(x) - \sigma_1(y)| < |\sigma_1(x) - \overline{\sigma_1(y)}| = |\sigma_2(x) - \sigma_2(y)| = |\sigma_2(x-y)| ,$$

a contradiction. Therefore we must have $\sigma_1 = \sigma_2$. □

Corollary 3. *Let K be a number field, and let $\rho_1, \dots, \rho_r, \sigma_1, \bar\sigma_1, \dots, \sigma_s, \bar\sigma_s$ be the distinct embeddings of K into $\mathbb{C}$. Then the distinct archimedean places of K are represented by the pull-backs of the standard absolute value on $\mathbb{C}$ (or $\mathbb{R}$) via $\rho_1, \dots, \rho_r, \sigma_1, \dots, \sigma_s$.*

Proof. Immediate. □

Therefore, if L/K is an extension of number fields, then for any archimedean place v of K, the set of places of L lying over v is nonempty and finite. Moreover, by Corollary 8 in the earlier handout, all of these places (of L) are archimedean.

If K is a function field with constant field F, then there are no archimedean absolute values on K that restrict to the trivial absolute value on F, again by Corollary 8 of the earlier handout (using the fact that the trivial absolute value is non-archimedean).

This says all that we intend to say in this handout about archimedean places, so we now concentrate on non-archimedean places.

Lemma 4. *Let L/K be an algebraic field extension. Then the only absolute value on L that extends the trivial absolute value on K is the trivial absolute value on L. (In other words, Theorem II 4.8 also holds if the absolute value on K is trivial; note that every field K is complete with respect to the trivial absolute value.)*

Proof. Let $|\cdot|$ be an absolute value on L that extends the trivial absolute value on K. We shall show that $|\cdot|$ is also trivial on L.

Since the trivial absolute value on K is non-archimedean, the absolute value on L is also non-archimedean (by Corollary 8 of the earlier handout).

We now claim that $|\alpha| \le 1$ for all $\alpha \in L$. Indeed, let $\alpha \in L$ and write

$$\operatorname{Irr}_{\alpha,K}(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0 .$$

Then

$$
\begin{aligned}
|\alpha|^n &= |-a_{n-1}\alpha^{n-1} - a_{n-2}\alpha^{n-2} - \dots - a_0| \\
&\le \max\{|a_{n-1}||\alpha|^{n-1}, |a_{n-2}||\alpha|^{n-2}, \dots, |a_0|\} \\
&\le \max\{|\alpha|^{n-1}, 1\} ,
\end{aligned}
$$

which gives $|\alpha| \le 1$. But also, if $\alpha \ne 0$ then $|1/\alpha| \le 1$, so we must have $|\alpha| = 1$. Thus the absolute value on L must be trivial. □

We now consider the following result for extensions of valuations coming from nonzero primes of Dedekind rings.

Lemma 5. *Let L/K be a finite extension of number fields or of function fields over a constant field F. If K and L are number fields, then let $A = \mathcal{O}_K$ and $B = \mathcal{O}_L$; if they are function fields, then let A be a Dedekind ring of finite type over F whose fraction field is K, and let B be the integral closure of A in L. Note that in either case B is finite over A. Let $|\cdot|$ be an absolute value on L. Assume that there exist a nonzero prime $\mathfrak{p}$ of A and a constant $C > 1$ such that $|x| = C^{-\nu_{\mathfrak{p}}(x)}$ for all $x \in K^*$. Then there is a prime ideal $\mathfrak{q}$ of B lying over $\mathfrak{p}$ such that $|x| = C^{-\nu_{\mathfrak{q}}(x)/e}$ for all $x \in L^*$, where $e = e_{\mathfrak{q}/K}$ is the ramification index.*

Proof. This follows fairly easily from Exercise 6 on page 166 (which may appear on a future homework assignment). Here we give a different proof, which applies only to this case.

First consider the special case in which L/K is normal.

Let $w\colon L^* \to \mathbb{R}$ be the valuation on L defined by $w(x) = -\log|x|$, and let $v = w\big|_K$ be its restriction to K. Note that $v(x) = \nu_{\mathfrak{p}}(x)\log C$ for all $x \in K^*$.

Fix an algebraic closure $\overline{L}_w$ of L_w (hence of K_v).

Choose a prime $\mathfrak{q}_0$ of B lying over $\mathfrak{p}$, and let $w_0(x) = (\nu_{\mathfrak{q}_0}(x)/e_{\mathfrak{q}_0/K})\log C$ for all $x \in L^*$. This is an extension of v to L, so by II Thm. 8.1 (i), there is an embedding $\tau\colon L \to \overline{L}_w$ over K such that $w_0 = w \circ \tau$.

Since L is normal over K, the image of τ equals the image of the canonical injection of L into L_w, so τ is an element of $\operatorname{Gal}(L/K)$. Let $\mathfrak{q} = \tau(\mathfrak{q}_0)$. Since τ is an automorphism of L over K, we have $e_{\mathfrak{q}_0/K} = e_{\mathfrak{q}/K} = e$, and therefore

$$
-\log|x| = w(x) = w_0(\tau^{-1}(x)) = \frac{\nu_{\mathfrak{q}_0}(\tau^{-1}(x))}{e}\log C = \frac{\nu_{\tau(\mathfrak{q}_0)}(x)}{e}\log C = \frac{\nu_{\mathfrak{q}}(x)}{e}\log C
$$

for all $x \in L^*$, as was to be shown.

For the general case, let L' be a finite normal extension of K containing L, choose an extension of $|\cdot|$ to L', let B' be the integral closure of B in L', and let $\mathfrak{q}'$ be a nonzero prime of B' that satisfies the conditions of the lemma for L'/K. Then the lemma for L/K follows by letting $\mathfrak{q} = \mathfrak{q}' \cap L$ and applying multiplicativity of the ramification index in towers. □

We are now able to prove the following generalization of II Prop. 3.7. (This result is somewhat academic, however, since we usually take the set of places of a number field to be a given.)

Theorem 6. *Let K be a number field, and let $|\cdot|$ be a nontrivial non-archimedean absolute value on K. Then there exist a prime $\mathfrak{p}$ of K and a constant $C > 1$ such that $|x| = C^{-v_{\mathfrak{p}}(x)}$ for all $x \in K^*$. In particular, this defines a canonical bijection between the set of non-archimedean places of K and the set of primes of K.*

Proof. By Lemma 4, the restriction of $|\cdot|$ to $\mathbb{Q}$ is nontrivial, so by II Prop. 3.7 it is equivalent to $|\cdot|_p$ for some rational prime p. By Lemma 5, there exist a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ lying over p and $C > 1$ that satisfy the condition of the theorem. Clearly any two distinct primes $\mathfrak{p}$ of K give inequivalent absolute values, hence distinct places, and this gives the bijection. $\square$

A similar result holds in the function field case, but proving it requires a result similar to II Prop. 3.7.

Lemma 7. *Let F be an arbitrary field, let $K = F(t)$ (with t an indeterminate), and let $|\cdot|$ be a nontrivial non-archimedean absolute value on K whose restriction to F is trivial. Then there exist a prime $\mathfrak{p}$ of either $F[t]$ or $F[1/t]$ (or both) and a constant $C > 1$ such that $|x| = C^{-v_{\mathfrak{p}}(x)}$ for all $x \in K^*$.*

Proof. After replacing t with $1/t$ if necessary, we may assume that $|t| \le 1$. Let $A = F[t]$. Then $|x| \le 1$ for all $x \in A$. Let $\mathfrak{p} = \{x \in A : |x| < 1\}$. Then (since $|\cdot|$ is non-archimedean) $\mathfrak{p}$ is a prime ideal in A. Also $\mathfrak{p} \ne (0)$ (because $|\cdot|$ is nontrivial and A generates K). Therefore $\mathfrak{p} = (\pi)$ for some irreducible $\pi \in F[t]$. We then have $|x| = |\pi|^{v_{\mathfrak{p}}(x)}$ for all nonzero $x \in A$, and therefore the same equation holds for all $x \in K^*$. $\square$

Theorem 8. *Let K be a function field with constant field F, and let $t \in K$ be transcendental over F (so that K is finite over $F(t)$). Let $|\cdot|$ be a nontrivial absolute value on K which is trivial on F. Then there exist (i) a nonzero prime ideal $\mathfrak{p}$ in the integral closure of either $F[t]$ or $F[1/t]$ in K and (ii) a constant $C > 1$, such that $|x| = C^{-v_{\mathfrak{p}}(x)}$ for all $x \in K^*$. In particular, this defines a canonical bijection between the set of places of K trivial on F and the set of closed points on the nonsingular projective curve over F with function field K.*

Proof. By Lemma 4, the restriction of $|\cdot|$ to $F(t)$ is nontrivial, so by Lemma 7 there exist a prime ideal $\mathfrak{p}_0$ of A, with A equal to either $F[t]$ or $F[1/t]$, and $C_0 > 1$ such that $|x| = C_0^{-v_{\mathfrak{p}_0}(x)}$ for all $x \in F(t)^*$. Let B be the integral closure of A in K. Then, by Lemma 5, there is a prime ideal $\mathfrak{p}$ of B such that $|x| = C^{-v_{\mathfrak{p}}(x)}$ for all $x \in K^*$, where $C = C_0^{1/e}$ and $e = e_{\mathfrak{p}/\mathfrak{p}_0}$ is the ramification index. Now (please forgive the algebraic geometry) $\operatorname{Spec} B$ is an open affine subset of the curve mentioned in the statement of the theorem, and $\mathfrak{p}$ corresponds to a closed point of $\operatorname{Spec} B$. With some additional work in algebraic geometry, we obtain the required bijection. $\square$