Math 254A. Valued Rings and Valued Fields

This handout gives some basic definitions and results concerning valued rings and valued fields.

Definition 1. A **valued ring** is an ordered pair $(R, |\cdot|)$, where R is a ring and $|\cdot|$ is an absolute value on R. A **homomomorphism** $\phi\colon (R, |\cdot|_R) \to (S, |\cdot|_S)$ of valued rings is a ring homomorphism $\phi\colon R \to S$ such that $|\phi(r)|_S = |r|_R$ for all $r \in R$.

Note that all homomorphisms $\phi\colon (R, |\cdot|_R) \to (S, |\cdot|_S)$ of valued rings are injective, since if $r \in \ker\phi$ then $|r|_R = |\phi(r)|_S = 0$, which implies $r = 0$. As such, we may at times use the word **embedding** instead of homomorphism.

Definition 2. A valued ring $(R, |\cdot|)$ is **archimedean** or **non-archimedean** if its absolute value is archimedean or non-archimedean, respectively.

A valued ring $(R, |\cdot|)$ gives rise to a metric space, with metric $d(x, y) = |x - y|$. This in turn, provides a topology on R.

Definition 3. A valued ring $(R, |\cdot|)$ is **complete** if its associated metric space is complete (i.e., all Cauchy sequences in R converge to a limit in R).

Definition 4. A **valued field** is a valued ring $(K, |\cdot|)$ in which K is a field. A **homomorphism** of valued fields is a homomorphism of valued rings (whose domain and codomain are valued fields). A valued field is **archimedean**, **non-archimedean**, or **complete** if it has that property as a valued ring.

From now on, we will often write $(R, |\cdot|)$ as just R.

Theorem 5. *Let $R = (R, |\cdot|)$ be a valued ring.*

(a). *The valued ring R can be embedded as a dense subring of a complete valued ring $(\widehat{R}, |\cdot|^{\widehat{\ }})$.*

(b). *Embeddings $R \to \widehat{R}$ of a valued ring R into a complete valued ring $\widehat{R}$ with dense image satisfy the following universal property. Every homomorphism $\psi\colon R \to S$ of valued rings, with S complete, factors uniquely through a homomorphism $\widehat{\psi}\colon \widehat{R} \to S$ of valued rings. As a consequence, embeddings $R \to \widehat{R}$, with $\widehat{R}$ complete and with dense image, are unique up to unique isomorphism.*

Proof. If R is the trivial ring, then the result is also trivial (and is left as an exercise). Therefore we assume from now on that $R \ne (0)$.

(a). Let $R_0 = R^{\mathbb{N}}$ be the ring $R \times R \times \dots$ of $\mathbb{N}$-indexed sequences in R, and let $R_1 \subseteq R_0$ be the subset of Cauchy sequences.

Claim. R_1 is a subring of R_0.

Proof. A sum of two Cauchy sequences is Cauchy, and so is the negative of a Cauchy sequence; therefore R_1 is an additive subgroup of R_0. The constant sequence $(1, 1, \dots)$

is Cauchy; therefore R_1 contains a unity element. Finally, the fact that R_1 is closed under multiplication follows from (i) all Cauchy sequences are bounded, and (ii)

$$|a_n b_n - a_m b_m| \le |a_n||b_n - b_m| + |b_m||a_n - a_m| \;.$$

This proves the claim. □

Claim. Let $\mathfrak{q} = \{(a_n) \in R_1 : |a_n| \to 0 \text{ as } n \to \infty\}$. Then $\mathfrak{q}$ is a prime ideal in R_1.

Proof. It is easy to see that $\mathfrak{q}$ is an additive subgroup of R_1. To see that it is an ideal, let $a = (a_n)_{n\in\mathbb{N}} \in \mathfrak{q}$ and $b = (b_n)_{n\in\mathbb{N}} \in R_1$. Then the sequence $(a_n b_n)$ converges to zero because the sequence $(|b_n|)$ is bounded and $a_n \to 0$. Therefore $\mathfrak{q}$ is an ideal in R_1.

Finally, the ideal $\mathfrak{q}$ is prime because the sequence $(1, 1, \dots)$ is not in $\mathfrak{q}$ (trivially), and if $(a_n), (b_n) \notin q$ then the sequences $(|a_n|)$ and $(|b_n|)$ converge to nonzero real numbers, and therefore so does the sequence $(|a_n b_n|)$. □

Now let $\widehat{R} = R_1/\mathfrak{q}$. It is an entire ring.

Let $a = (a_n) \in R_1$. Then $\lim_{n\to\infty} |a_n|$ converges (by the triangle inequality the sequence $(|a_n|)$ is a Cauchy sequence of real numbers). Also, if $a = (a_n)$ and $b = (b_n)$ are in R_1 and $a - b \in \mathfrak{q}$, then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |b_n|$, so $|a|^\frown := \lim_{n\to\infty} |a_n|$ is a well-defined function on $\widehat{R}$. It is easy to check that this satisfies the conditions for an absolute value on $\widehat{R}$, so $(\widehat{R}, |\cdot|^\frown)$ is a valued ring.

Since constant sequences are Cauchy, $a \mapsto (a, a, \dots)$ is a well-defined ring homomorphism $\phi_0 \colon R \to R_1$. Let $\phi \colon R \to \widehat{R}$ be the map corresponding to ϕ_0. Then it is clear that $|\phi(a)|^\frown = |a|$ for all $a \in R$, so ϕ is a homomorphism of valued rings.

Next, the image of ϕ is dense in $\widehat{R}$. Indeed, let $\alpha \in \widehat{R}$ be any element. We need to show that for all $\epsilon > 0$ there is an $a = a_\epsilon \in R$ such that $|\alpha - \phi(a)|^\frown \le \epsilon$. To see this, lift α to a Cauchy sequence $(a_0, a_1, \dots) \in R_1$. Pick $N = N_\epsilon \in \mathbb{N}$ such that $|a_i - a_j| \le \epsilon$ for all $i, j \ge N$. In particular, $|a_i - a_N| \le \epsilon$ for all $i \ge N$, so $|\alpha - \phi(a_N)|^\frown = \lim_{i\to\infty} |a_i - a_N| \le \epsilon$, as was to be shown.

Finally, we claim that $\widehat{R}$ is complete. Indeed, let $(\alpha_0, \alpha_1, \dots)$ be a Cauchy sequence in $\widehat{R}$. Given $n \in \mathbb{N}$, pick $N_n \in \mathbb{N}$ such that $|\alpha_i - \alpha_j|^\frown \le 2^{-n}$ for all $i, j \ge N_n$, and pick $a_n \in R$ such that $|\alpha_{N_n} - \phi(a_n)|^\frown \le 2^{-n}$. Then the sequence $(a_0, a_1, \dots)$ is Cauchy, because for all $n \le m$ in $\mathbb{N}$,

$$\begin{aligned} |a_n - a_m| = |\phi(a_n) - \phi(a_m)|^\frown &\le |\phi(a_n) - \alpha_{N_n}|^\frown + |\alpha_{N_n} - \alpha_{N_m}|^\frown + |\alpha_{N_m} - \phi(a_m)|^\frown \\ &\le 2^{-n} + 2^{-n} + 2^{-m} \\ &\le 3 \cdot 2^{-n} \;. \end{aligned}$$

Therefore this sequence defines an element $\alpha \in \widehat{R}$. Also $\lim_{i\to\infty} \alpha_i = \alpha$ (in $\widehat{R}$), because for all $n \ge 0$ and all $i \ge \max\{n, N_n\}$,

$$\begin{aligned} |\alpha_i - \alpha|^\frown &\le |\alpha_i - \alpha_{N_m}|^\frown + |\alpha_{N_m} - \phi(a_n)|^\frown + \lim_{j\to\infty} \left(|\phi(a_n) - \phi(a_j)|^\frown + |\phi(a_j) - \alpha)|^\frown \right) \\ &\le 2^{-n} + 2^{-n} + \lim_{j\to\infty} |\phi(a_n) - \phi(a_j)|^\frown + \lim_{j\to\infty} |\phi(a_j) - \alpha)|^\frown \\ &\le 2^{-n} + 2^{-n} + 3 \cdot 2^{-n} + 0 = 5 \cdot 2^{-n} \;. \end{aligned}$$

Thus $(\widehat{R}, |\cdot|^\frown)$ is complete, so (a) is proved.

(b). First of all, note that all homomorphisms of valued rings are continuous (relative to the topologies on their domains and codomains defined by the absolute value).

Now let $\phi\colon R \to \widehat{R}$ and $\psi\colon R \to S$ be homomorphisms of valued rings. Assume that $\widehat{R}$ and S are complete and that ϕ has dense image.

We first construct $\widehat{\psi}$. Let $\alpha \in \widehat{R}$. Since ϕ has dense image, there is a sequence $(a_0, a_1, \dots)$ of elements of R such that $(\phi(a_0), \phi(a_1), \dots)$ converges to α. The latter sequence is Cauchy, and therefore so are the sequences $(a_0, a_1, \dots)$ and $(\psi(a_0), \psi(a_1), \dots)$ (in R and S, respectively). Since S is complete, the latter sequence converges to a limit $\beta \in S$.

We claim that this element β is independent of the choice of $(a_0, a_1, \dots)$. Indeed, let $(b_0, b_1, \dots)$ be another sequence of elements of R such that $(\phi(b_0), \phi(b_1), \dots)$ converges to α. Then $\lim_{i\to\infty} |\phi(a_i) - \phi(b_i)| = 0$, which implies $\lim_{i\to\infty} |a_i - b_i| = 0$, and therefore $\lim_{i\to\infty} |\psi(a_i) - \psi(b_i)| = 0$. Therefore $\lim_{i\to\infty} \psi(b_i) = \beta$ also.

Thus, we let $\widehat{\psi}(\alpha) = \beta$. It is easy to check that ψ is a ring homomorphism. For example, let $\alpha, \beta \in \widehat{R}$, and let $(a_0, a_1, \dots)$ and $(b_0, b_1, \dots)$ be sequences of elements of R such that $(\phi(a_0), \phi(a_1), \dots)$ and $(\phi(b_0), \phi(b_1), \dots)$ converge to α and β, respectively. Then $(\phi(a_0 b_0), \phi(a_1 b_1), \dots)$ converges to $\alpha\beta$, and

$$\begin{aligned}\widehat{\psi}(\alpha\beta) &= \lim_{n\to\infty} \psi(a_n b_n) = \lim_{n\to\infty} \psi(a_n)\psi(b_n) \\ &= \left(\lim_{n\to\infty} \psi(a_n)\right)\left(\lim_{n\to\infty} \psi(b_n)\right) = \widehat{\psi}(\alpha)\widehat{\psi}(\beta) .\end{aligned}$$

We then have that $\psi = \widehat{\psi} \circ \phi$, because the constant sequence $(\phi(a), \phi(a), \dots)$ converges to $\phi(a)$, and therefore

$$\widehat{\psi}(\phi(a)) = \lim_{i\to\infty} \psi(a) = \psi(a) \qquad \text{for all } a \in R .$$

Finally, to show uniqueness, suppose that $\widetilde{\psi}\colon \widehat{R} \to S$ is another homomorphism of valued rings such that $\widetilde{\psi} \circ \phi = \psi$. Then $\widehat{\psi}$ and $\widetilde{\psi}$ are continuous functions from $\widehat{R}$ to S. Since they coincide on the dense subset $\phi(R)$ of $\widehat{R}$, they must be equal. □

Proposition 6. *If K is a valued field, then $\widehat{K}$ is also a (valued) field.*

Proof. Let $\alpha \in K^*$, and let $(a_0, a_1, \dots)$ be a Cauchy sequence in K that converges to α. There is some N such that $|a_i - \alpha| < |\alpha|$ for all $i \ge N$, and therefore the sequence $(1/a_N, 1/a_{N+1}, \dots)$ is a Cauchy sequence in K whose limit $\beta \in \widehat{K}$ satisfies $\alpha\beta = 1$. □

Proposition 7. *Let $R = (R, |\cdot|)$ be a valued ring. Then the following are equivalent.*

(i). *R is non-archimedean;*
(ii). *$|n| \le 1$ for all $n \in \mathbb{N}$; and*
(iii). *the set $\{|n| : n \in \mathbb{N}\}$ is bounded.*

Proof. (i) $\Longrightarrow$ (ii) is true by induction, and (ii) $\Longrightarrow$ (iii) is trivial. For (iii) $\Longrightarrow$ (i), assume that $|n| \le L$ for all $n \in \mathbb{N}$. Then, for all $n \ge 0$ and all $x, y \in R$,

$$|x+y|^n = \left| \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \right| \le \sum_{i=0}^{n} \left| \binom{n}{i} \right| \max\{|x|, |y|\}^n \le (n+1)L \max\{|x|, |y|\}^n \ .$$

Taking n large and noting that $\lim\limits_{n\to\infty} \sqrt[n]{(n+1)L} = 1$ then gives $|x+y| \le \max\{|x|, |y|\}$, so $|\cdot|$ is non-archimedean. $\square$

Corollary 8. *Let $\phi\colon R \to S$ be a homomorphism of valued rings. If R is archimedean, or non-archimedean, then S has that same property.*

Proof. Indeed, this follows from the equivalence (i) $\Longleftrightarrow$ (iii) of Proposition 7. $\square$