Math 254A. $\zeta'(0)$

or

The Product of the Positive Integers

We will evaluate $\zeta'(0)$, using the functional equation

$$\zeta(s)=\frac{\pi^{(s-1)/2}\Gamma\bigl(\frac{1-s}{2}\bigr)\zeta(1-s)}{\pi^{-s/2}\Gamma\bigl(\frac{s}{2}\bigr)}\ . \tag{1}$$

First, let

$$\xi(s)=s\zeta(1-s)\ ,$$

so that ξ is an entire function; then $\xi(0)=-1$. Moreover, the **Kronecker limit formula** says that

$$\zeta(s)=\frac{1}{s-1}+\gamma+O(|s-1|)$$

near $s=1$, where γ is Euler's constant

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)=0.5772\dots\ .$$

Thus we have

$$\xi(s)=-1+\gamma s+O(|s|)\qquad\text{near }\ s=0\ . \tag{2}$$

Rewriting (1) with the definition of ξ gives

$$\zeta(s)=\frac{\pi^{s-1/2}\Gamma\bigl(\frac{1-s}{2}\bigr)\xi(s)}{2\Gamma\bigl(\frac{s}{2}+1\bigr)}\ .$$

One can plug in $s=0$ and the fact that $\Gamma\bigl(\frac{1}{2}\bigr)=\sqrt{\pi}$ to again obtain the fact that $\zeta(0)=-\frac{1}{2}$. Now take logarithmic derivatives at $s=0$:

$$\frac{\zeta'(0)}{\zeta(0)}=\log\pi-\frac{1}{2}\cdot\frac{\Gamma'\bigl(\frac{1}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}\bigr)}+\frac{\xi'(0)}{\xi(0)}-\frac{1}{2}\cdot\frac{\Gamma'(1)}{\Gamma(1)}\ . \tag{3}$$

We already know from (2) that

$$\frac{\xi'(0)}{\xi(0)}=-\gamma\ .$$

To compute $\Gamma'(1)/\Gamma(1)$, recall the defining integral for the gamma function:

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt\ .$$

Differentiating under the integral sign and substituting $s = 1$ gives

$$\Gamma'(1) = \int_0^\infty e^{-t}(\log t)dt \ .$$

Looking this up in a table of definite integrals yields the value $-\gamma$; thus

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma \ .$$

Finally, Legendre's duplication formula (Neukirch VIII Prop. 1.2) says that

$$\Gamma(s)\Gamma(s+\tfrac{1}{2}) = \frac{2\sqrt{\pi}}{2^{2s}}\Gamma(2s) \ .$$

Taking logarithmic derivatives at $s = \frac{1}{2}$ again gives the relation

$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + \frac{\Gamma'(1)}{\Gamma(1)} = -2\log 2 + 2\frac{\Gamma'(1)}{\Gamma(1)} \ ;$$
$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = -\gamma - 2\log 2 \ .$$

Putting these values back into (3) then gives

$$\frac{\zeta'(0)}{\zeta(0)} = \log\pi + \log 2;$$
$$\zeta'(0) = -\frac{1}{2}\log 2\pi \ .$$

Since

$$\frac{d}{ds}\left(\frac{1}{n^s}\right)\Big|_{s=0} = -\log n \cdot n^{-s}\Big|_{s=0} = -\log n \ ,$$

it follows that

$$\zeta'(0) \text{ “=” } \sum_{n=1}^{\infty} -\log n$$

and therefore

$$\prod_{n=1}^{\infty} n \text{ “=” } \sqrt{2\pi} \ .$$