Math 254A. The Discriminant of $x^n + ax + b$

The discriminant of a polynomial f of degree n is a homogeneous polynomial of degree $n(n-1)$ in the roots of f. By Lang's *Algebra,* Ch. IV Thm. 6.1, this discriminant is also a polynomial in the coefficients a and b of f. These coefficients are homogeneous polynomials of degrees $n-1$ and n in the roots of f, respectively, so the discriminant is a weighted homogeneous polynomial ϕ of degree $n(n-1)$ in a and b, where a has weight $n-1$ and b has weight n. Therefore any monomial $a^i b^j$ occurring in ϕ must have $(n-1)i+nj = n(n-1)$. There are only two possibilities with $i \geq 0$ and $j \geq 0$, namely $(i,j) = (0, n-1)$ and $(i,j) = (n, 0)$, so the discriminant is of the form

$$D = \alpha a^n + \beta b^{n-1} \ . \tag{*}$$

We first restrict to working over $\mathbb{C}$.

We start by finding the coefficient β. This can be found by computing the discriminant of the polynomial $x^n - 1$, which has roots $1, \zeta, \ldots, \zeta^{n-1}$, where ζ is a primitive n^{th} root of unity. We have

$$\begin{aligned} D_{x^n-1} &= \prod_{0 \leq i < j < n} (\zeta^i - \zeta^j)^2 \\ &= (-1)^{n(n-1)/2} \prod_{i=0}^{n-1} \prod_{\substack{0 \leq j < n \\ j \neq i}} (\zeta^i - \zeta^j) \\ &= (-1)^{n(n-1)/2} \prod_{i=0}^{n-1} \prod_{\ell=1}^{n-1} (\zeta^i - \zeta^{i+\ell}) \\ &= (-1)^{n(n-1)/2} \left(\prod_{i=0}^{n-1} \zeta^{i(n-1)} \right) \left(\prod_{\ell=1}^{n-1} (1 - \zeta^\ell) \right)^n . \end{aligned}$$

The first product in this expression is $\zeta^{n(n-1)^2/2}$. If n is odd, then $(n-1)^2/2 \in \mathbb{Z}$, and the product is 1; if n is even, then this is an odd power of $\zeta^{n/2} = -1$. Thus, the first product is $(-1)^{n-1}$. The second product equals n, because it is the value of the polynomial

$$\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + 1 \ ,$$

evaluated at $x = 1$. Thus

$$D_{x^n-1} = (-1)^{n(n-1)/2} (-1)^{n-1} n^n \ ,$$

and since this equals $(-1)^{n-1}\beta$, we have

$$\beta = (-1)^{n(n-1)/2} n^n \ .$$

Next we find the coefficient α, this time using the polynomial $x^n - x$. This polynomial has roots 0, ζ^0, ζ^1, ..., ζ^{n-2}, where ζ is now a primitive $(n-1)^{\text{st}}$ root of unity. The discriminant of this polynomial is therefore

$$\begin{aligned} D_{x^n-x} &= \prod_{i=0}^{n-2}(\zeta^i - 0)^2 \cdot \prod_{\substack{0\le i<n-1\\ 0\le j<n-1\\ i<j}} (\zeta^i - \zeta^j)^2 \\ &= \zeta^{(n-1)(n-2)} \cdot (\text{discriminant of } x^{n-1}-1) \\ &= (-1)^{(n-1)(n-2)/2}(-1)^{n-2}(n-1)^{n-1} \\ &= -(-1)^{n(n-1)/2}(n-1)^{n-1} \ . \end{aligned}$$

This equals $(-1)^n\alpha$, so

$$\alpha = (-1)^{n(n-1)/2}(1-n)^{n-1} \ .$$

By (*), we then have

$$D = (-1)^{n(n-1)/2}\big((1-n)^{n-1}a^n + n^n b^{n-1}\big) \ . \qquad (**)$$

The above computation determined the discriminant for polynomials of the above form over $\mathbb{C}$. But the discriminant of a polynomial of the form $x^n + ax + b$ over an arbitrary commutative ring A is of the form $\phi(a,b)$ for some polynomial $\phi \in A[t,u]$; moreover, by functoriality, if $\alpha\colon A \to B$ is a homomorphism of commutative rings, then the polynomials $\phi_A(t,u)$ and $\phi_B(t,u)$ giving discriminants of polynomials of the given form with coefficients in A and B, respectively, are related by $\phi_B = \phi_A^\alpha$. Thus, (**) gives the discriminant for polynomials $x^n + ax + b$ in $\mathbb{Z}[x]$ and (via the unique map $\mathbb{Z} \to A$) it also gives the discriminant for arbitrary commutative rings A.