Math 254A. The Discriminant of \( x^n + ax + b \)

The discriminant of a polynomial \( f \) of degree \( n \) is a homogeneous polynomial of degree \( n(n-1) \) in the roots of \( f \). By Lang’s Algebra, Ch. IV Thm. 6.1, this discriminant is also a polynomial in the coefficients \( a, b \) of \( f \). These coefficients are homogeneous polynomials of degrees \( n-1 \) and \( n \) in the roots of \( f \), so the discriminant is a weighted homogeneous polynomial \( \phi \) of degree \( n(n-1) \) in \( a \) and \( b \), where \( a \) has weight \( n-1 \) and \( b \) has weight \( n \). Therefore any monomial \( a^i b^j \) occurring in \( \phi \) must have \( (n-1)i + nj = n(n-1) \). There are only two possibilities with \( i \geq 0 \) and \( j \geq 0 \), namely \( (i,j) = (0,n-1) \) and \( (i,j) = (n,0) \), so the discriminant is of the form

\[
D = \alpha a^n + \beta b^{n-1} .
\] (*)

We first restrict to working over \( \mathbb{C} \).

We start by finding the coefficient \( \beta \). This can be found by computing the discriminant of the polynomial \( x^n - 1 \), which has roots \( 1, \zeta, \ldots, \zeta^{n-1} \), where \( \zeta \) is a primitive \( n \)th root of unity. We have

\[
D_{x^n-1} = \prod_{0 \leq i < j < n} (\zeta^i - \zeta^j)^2
\]

\[
= (-1)^{n(n-1)/2} \prod_{i=0}^{n-1} \prod_{0 \leq j < n \atop j \neq i} (\zeta^i - \zeta^j)
\]

\[
= (-1)^{n(n-1)/2} \prod_{i=0}^{n-1} \prod_{\ell=1}^{n-1-i} (\zeta^i - \zeta^{i+\ell})
\]

\[
= (-1)^{n(n-1)/2} \left( \prod_{i=0}^{n-1} \zeta^{i(n-1)} \right) \left( \prod_{\ell=1}^{n-1} (1 - \zeta^\ell) \right)^n .
\]

The first product in this expression is \( \zeta^{n(n-1)/2} \). If \( n \) is odd, then \( (n-1)/2 \in \mathbb{Z} \), and the product is 1; if \( n \) is even, then this is an odd power of \( \zeta^{n/2} = -1 \). Thus, the first product is \( (-1)^{n-1} \). The second product equals \( n \), because it is the value of the polynomial

\[
\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + 1 ,
\]

evaluated at \( x = 1 \). Thus

\[
D_{x^n-1} = (-1)^{n(n-1)/2} (-1)^{n-1} n^n ,
\]

and since this equals \( (-1)^{n-1} \beta \), we have

\[
\beta = (-1)^{n(n-1)/2} n^n .
\]
Next we find the coefficient $\alpha$, this time using the polynomial $x^n - x$. This polynomial has roots $0, \zeta^0, \zeta^1, \ldots, \zeta^{n-2}$, where $\zeta$ is now a primitive $(n-1)^{\text{st}}$ root of unity. The discriminant of this polynomial is therefore

$$D_{x^n - x} = \prod_{i=0}^{n-2} (\zeta^i - 0)^2 \cdot \prod_{\substack{0 \leq i < n-1 \leq \zeta^i < \zeta^j}} (\zeta^i - \zeta^j)^2$$

$$= \zeta^{(n-1)(n-2)} \cdot \text{(discriminant of } x^{n-1} - 1)$$

$$= (-1)^{(n-1)(n-2)/2}(-1)^{n-2}(n-1)^{n-1}$$

$$= -(-1)^{n(n-1)/2(n-1)^{n-1}}.$$

This equals $(-1)^n \alpha$, so

$$\alpha = (-1)^{n(n-1)/2(1-n)^{n-1}}.$$

By (*), we then have

$$D = (-1)^{n(n-1)/2}((1-n)^{n-1}a^n + n^n b^{n-1}). \quad (** \text{) }$$

The above computation determined the discriminant for polynomials of the above form over $\mathbb{C}$. But the discriminant of a polynomial of the form $x^n + ax + b$ over an arbitrary commutative ring $A$ is of the form $\phi(a, b)$ for some polynomial $\phi \in A[t, u]$; moreover, by functoriality, if $\alpha: A \rightarrow B$ is a homomorphism of commutative rings, then the polynomials $\phi_A(t, u)$ and $\phi_B(t, u)$ giving discriminants of polynomials of the given form with coefficients in $A$ and $B$, respectively, are related by $\phi_B = \phi_A^\alpha$. Thus, (** \text{) gives the discriminant for polynomials } x^n + ax + b \text{ in } \mathbb{Z}[x] \text{ and (via the unique map } \mathbb{Z} \rightarrow A \) it also gives the discriminant for arbitrary commutative rings $A$. 