1. Give (separate) examples of a Noetherian ring $R$, a Noetherian $R$-module $M$, and a primary decomposition of a submodule $M'$ of $M$ such that:
   (a). Some $M_i$ is $P_i$-primary with $P_i \notin \text{Ass}(M/M')$;
   (b). The primary decomposition is irredundant but not minimal (hint: $k[x,y]$).

2. Let $k$ be a field. Let
   
   $R = \left\{ \sum_{i,j \in \mathbb{N}} a_{ij} x^i y^j : a_{01} = 0 \right\}$

   be the ring given in class on Friday, 6 March, with ideals

   $P = (xy, y^2, y^3)$ and $I = P^2 = (x^2 y^2, xy^4, y^5)$.

   Recall that $\sqrt{I} = P$ is prime but that $I$ is not primary. The object of this problem is to compute a primary decomposition for $I$ in $R$.
   (a). Prove that $(0) \notin \text{Ass}_R(R/I)$.
   (b). Prove that $P$ is minimal over $\text{Ann}(R/I)$.
   (c). Find $m \in R/I$ such that $\text{Ann} m = P$.
   (d). Find the $P$-primary part of 0 in $R/I$.
   (e). Prove that $\text{Ass}(R/I) = \{P, M_1, \ldots, M_r\}$, where $M_1, \ldots, M_r$ are maximal ideals in $R$, with $r \geq 1$. For this part you may assume without proof that all prime ideals of $R$ strictly containing $P$ are maximal.
   (f). Finish finding a primary decomposition of $I$.