Math 115. Slides from the Lecture of December 3

Periodic Continued Fractions (continued)

Lemma. *Let* $\xi \in \mathbb{R}$, $k \in \mathbb{Z}$, *and* $\theta = \xi + k$. *Then*

(a). $\theta - [\theta] = \xi - [\xi]$, *and*

(b). *If* $\xi = \langle a_0, a_1, a_2, \dots \rangle$, *then* $\theta = \langle a_0 + k, a_1, a_2, \dots \rangle$.

Proof. Part (a) follows easily from the identity $[\xi + k] = [\xi] + k$.

Part (b) follows from the above identity and part (a), since ξ_1 is the same for both ξ and θ:

$$\frac{1}{\xi - [\xi]} = \frac{1}{\theta - [\theta]} .$$

□

Last time: Let d be a positive integer, not a perfect square. Then $\sqrt{d} + [\sqrt{d}]$ is purely periodic.

Also recall that $\xi_i = \dfrac{m_i + \sqrt{d}}{q_i}$.

Proposition. *Write*

$$\sqrt{d} + \left[\sqrt{d}\right] = \langle \overline{a_0, a_1, \dots, a_{r-1}} \rangle ,$$

with r *minimal. Then* $q_i \neq 1$ *for all* $0 < i < r$, *and* $q_i = -1$ *never occurs.*

Proof. If $q_i = 1$ then $\xi_i = m_i + \sqrt{d}$, so by part (a) of the earlier lemma,

$$\xi_{i+1} = \frac{1}{\xi_i - [\xi_i]} = \frac{1}{m_i + \sqrt{d} - [m_i + \sqrt{d}]} = \frac{1}{[\sqrt{d}] + \sqrt{d} - [[\sqrt{d}] + \sqrt{d}]} = \xi_1 .$$

Therefore, going $r - 1$ steps forward gives

$$\xi_i = \xi_{i+r} = \xi_{i+1+(r-1)} = \xi_{1+(r-1)} = \xi_r = \sqrt{d} + [\sqrt{d}] .$$

But if $0 < i < r$, then we would have $\sqrt{d} + \left[\sqrt{d}\right] = \langle \overline{a_0, a_1, \dots, a_{i-1}} \rangle$, with $i < r$, contradicting the choice of r.

As for $q_i = -1$, all of the ξ_i are purely periodic, so $q_i < 0$ would imply $\xi_i > \xi_i'$, contradicting Theorem 7.20. □

Since $q_0 = q_r = 1$, we have that $q_i = 1$ if and only if i is a multiple of r.

Now note that $a_0 = [\sqrt{d} + [\sqrt{d}]] = [\sqrt{d}] + [\sqrt{d}] = 2[\sqrt{d}]$. So, replacing a_0 with $[\sqrt{d}]$, we have

$$\sqrt{d} + [\sqrt{d}] = \langle \overline{2a_0, a_1, a_2, \dots, a_{r-1}} \rangle = \langle 2a_0, \overline{a_1, a_2, \dots, a_{r-1}, 2a_0} \rangle$$

and therefore, by the lemma,

$$\sqrt{d} = \langle a_0, \overline{a_1, a_2, \dots, a_{r-1}, 2a_0} \rangle .$$

Again, we have $q_i = 1$ if and only if $r \mid i$, and $q_i = -1$ never occurs.

Pell's Equation (known already in ancient India)

Consider the equation

$$x^2 - dy^2 = N \ ,$$

where $d, N \in \mathbb{Z}$ are fixed and we are looking for solutions in integers x, y.

If $d \le 0$, then $|x| \le \sqrt{N}$, which is easy to solve.

If $d = m^2$ is a perfect square, then $x^2 - dy^2 = (x+my)(x-my)$, again leading to an equation which is easy to solve.

So from now on we assume that d is a positive integer, not a perfect square.

Finding all solutions with $x = 0$ or $y = 0$ is easy. All other solutions come in fours: $(\pm x, \pm y)$.

Therefore we concentrate on *positive* solutions, in which $x > 0$ and $y > 0$.

Theorem 7.21. *With h_i, k_i, ξ_i, q_i, m_i, and d as usual, we have*

$$h_{n-1}^2 - dk_{n-1}^2 = (-1)^n q_n \qquad \text{for all } n \ge 0 \ .$$

Proof (sketch). The equation

$$\sqrt{d} = \xi = \frac{\xi_n h_{n-1} + h_{n-2}}{\xi_n k_{n-1} + k_{n-2}} = \frac{(m_n + \sqrt{d})h_{n-1} + q_n h_{n-2}}{(m_n + \sqrt{d})k_{n-1} + q_n k_{n-2}}$$

leads to

$$\sqrt{d}(m_n k_{n-1} + q_n k_{n-2} - h_{n-1}) + (dk_{n-1} - m_n h_{n-1} + q_n h_{n-2}) = 0 \ .$$

Letting $A = m_n k_{n-1} + q_n k_{n-2} - h_{n-1}$ and $B = dk_{n-1} - m_n h_{n-1} + q_n h_{n-2}$, we see that $A = B = 0$ (since $A, B \in \mathbb{Z}$ and $\sqrt{d}$ is irrational). Eliminating m_n from these two equations gives

$$q_n(h_{n-1}k_{n-2} - k_{n-1}h_{n-2}) = h_{n-1}^2 - dk_{n-1}^2 \ ,$$

which gives the theorem because $h_{n-1}k_{n-2} - k_{n-1}h_{n-2} = (-1)^n$. □

Theorem. *If $|N| < \sqrt{d}$ then every positive solution to $x^2 - dy^2 = N$ equals (gh_n, gk_n) for some $g \in \mathbb{Z}_{>0}$, $n \ge 0$.*

Proof (even more sketchy). Reduce to Theorem 7.14, which says that if $a, b \in \mathbb{Z}_{>0}$ and $|\frac{a}{b} - \xi| < \frac{1}{2b^2}$, then a/b is a convergent. □