Approximation to Irrational Numbers

Throughout today’s class:
ξ is an irrational number and \( \langle a_0, a_1, \ldots \rangle \) is its simple continued fraction expansion.

Also:
\[
\begin{align*}
h_{-2} &= 0, & h_{-1} &= 1, & h_i &= a_i h_{i-1} + h_{i-2} \quad \text{for all } i \geq 0 \\
k_{-2} &= 1, & k_{-1} &= 0, & k_i &= a_i k_{i-1} + k_{i-2}
\end{align*}
\]

**Theorem 7.11.** For all \( n \geq 0 \),
\[
\left| \xi - \frac{h_n}{k_n} \right| < \frac{1}{k_n k_{n+1}} \quad \text{and} \quad \left| k_n \xi - h_n \right| < \frac{1}{k_{n+1}}.
\]

**Proof.** The first inequality holds because
\[
\left| \xi - \frac{h_n}{k_n} \right| = |\xi - r_n| < |r_{n+1} - r_n| = \frac{1}{k_n k_{n+1}}.
\]

The second inequality follows from the first by multiplying both sides by \( k_n \). \(\square\)

**Corollary.** \( \left| \xi - \frac{h_n}{k_n} \right| < \frac{1}{k_n^2} \) for all \( n \geq 0 \).

**Proof.** Indeed, \( k_{n+1} \geq k_n \) for all \( n \geq 0 \). \(\square\)

**Theorem 7.13.** Let \( a/b \) be a rational number (with \( a, b \in \mathbb{Z} \) and \( b > 0 \)).

(a). If \( n > 0 \) and \( \left| \xi - \frac{a}{b} \right| < |\xi - r_n| \) then \( b > k_n \)

(b). If \( n \geq 0 \) and \( |b \xi - a| < |k_n \xi - h_n| \) then \( b \geq k_{n+1} \).

**Proof.** We first show that (b) implies (a) by showing the contrapositive. Suppose (a) is false. Then:

(i) \( n > 0 \), (ii) \( \left| \xi - \frac{a}{b} \right| < |\xi - r_n| \), and (iii) \( b \leq k_n \).

Then (i) implies \( n \geq 0 \), and (ii) and (iii) imply:
\[
|b \xi - a| = b \left| \xi - \frac{a}{b} \right| < b |\xi - r_n| \leq k_n \left| \xi - \frac{h_n}{k_n} \right| = |k_n \xi - h_n|,
\]
yet \( b \leq k_n < k_{n+1} \) (by (iii) and (i)), so (b) is false.
To show (b), we prove the contrapositive. Assume that $n \geq 0$ and $b < k_{n+1}$.

First, we look for a rational number $r$ such that

$$\frac{h_{n+1}r + h_n}{k_{n+1}r + k_n} = f_{n+2}(r) = \frac{a}{b}.$$ 

Letting $r = y/x$, this leads to equations

$$\begin{align*}
  h_n x + h_{n+1} y &= a \\
  k_n x + k_{n+1} y &= b
\end{align*}$$

$$\begin{bmatrix} h_n & h_{n+1} \\
  k_n & k_{n+1} \end{bmatrix} \begin{bmatrix} x \\
  y \end{bmatrix} = \begin{bmatrix} a \\
  b \end{bmatrix}.$$ 

Let $M = \begin{bmatrix} h_n & h_{n+1} \\
  k_n & k_{n+1} \end{bmatrix}$. We know that det $M = \pm 1$, so $M^{-1}$ has integer entries, and

$$\begin{bmatrix} x \\
  y \end{bmatrix} = M^{-1} \begin{bmatrix} a \\
  b \end{bmatrix},$$

so there is a unique solution $(x, y)$, and $x, y \in \mathbb{Z}$.

Claim. $x \neq 0$, and $x > 0 \iff y \leq 0$.

Proof.

Case I. If $y \leq 0$, then $k_n x = b - k_{n+1}y \geq b > 0$, so $x > 0$.

Case II. If $y > 0$, then $b < k_{n+1} \leq k_{n+1} y$, so $k_n x = b - k_{n+1}y < 0$; therefore $x < 0$.

Now since $\xi$ is between $r_n$ and $r_{n+1}$, $k_n \xi - h_n$ and $k_{n+1} \xi - h_{n+1}$ have opposite signs. Therefore $x(k_n \xi - h_n)$ and $y(k_{n+1} \xi - h_{n+1})$ are either both $\geq 0$ or both $\leq 0$.

Adding them gives

$$x(k_n \xi - h_n) + y(k_{n+1} \xi - h_{n+1}) = (k_n x + k_{n+1} y) \xi - (h_n x + h_{n+1} y) = b \xi - a;$$

therefore

$$|b \xi - a| = |x(k_n \xi - h_n) + y(k_{n+1} \xi - h_{n+1})|$$

$$\geq |x(k_n \xi - h_n)| + |y(k_{n+1} \xi - h_{n+1})|$$

$$\geq |x(k_n \xi - h_n)|,$$

$$\geq |k_n \xi - h_n|.$$