Math 115. Slides from the Lecture of November 29

Approximation to Irrational Numbers

Throughout today's class:
ξ is an irrational number and $\langle a_0, a_1, \dots \rangle$ is its simple continued fraction expansion.

Also:

$$\begin{aligned} h_{-2} &= 0\,, \quad h_{-1} = 1\,, \quad h_i = a_i h_{i-1} + h_{i-2} \qquad \text{for all } i \geq 0 \\ k_{-2} &= 1\,, \quad k_{-1} = 0\,, \quad k_i = a_i k_{i-1} + k_{i-2} \end{aligned}$$

Theorem 7.11. *For all* $n \geq 0$,

$$\left|\xi - \frac{h_n}{k_n}\right| < \frac{1}{k_n k_{n+1}} \qquad \textit{and} \qquad \left|k_n\xi - h_n\right| < \frac{1}{k_{n+1}}\,.$$

Proof. The first inequality holds because

$$\left|\xi - \frac{h_n}{k_n}\right| = |\xi - r_n| < |r_{n+1} - r_n| = \frac{1}{k_n k_{n+1}}\,.$$

The second inequality follows from the first by multiplying both sides by k_n. □

Corollary. $\left|\xi - \dfrac{h_n}{k_n}\right| < \dfrac{1}{k_n^2}$ *for all* $n \geq 0$.

Proof. Indeed, $k_{n+1} \geq k_n$ for all $n \geq 0$. □

Theorem 7.13. *Let* a/b *be a rational number (with* $a, b \in \mathbb{Z}$ *and* $b > 0$*).*

(a). *If* $n > 0$ *and* $\left|\xi - \dfrac{a}{b}\right| < |\xi - r_n|$ *then* $b > k_n$

(b). *If* $n \geq 0$ *and* $|b\xi - a| < |k_n\xi - h_n|$ *then* $b \geq k_{n+1}$.

Proof. We first show that (b) implies (a) by showing the contrapositive. Suppose (a) is false. Then:

$$\text{(i)}\ \ n > 0\,, \qquad \text{(ii)}\ \ \left|\xi - \frac{a}{b}\right| < |\xi - r_n|\,, \quad \text{and} \quad \text{(iii)}\ \ b \leq k_n\,.$$

Then (i) implies $n \geq 0$, and (ii) and (iii) imply:

$$|b\xi - a| = b\left|\xi - \frac{a}{b}\right| < b|\xi - r_n| \leq k_n\left|\xi - \frac{h_n}{k_n}\right| = |k_n\xi - h_n|\,,$$

yet $b \leq k_n < k_{n+1}$ (by (iii) and (i)), so (b) is false.

To show (b), we prove the contrapositive. Assume that $n \geq 0$ and $b < k_{n+1}$.

First, we look for a rational number r such that

$$\frac{h_{n+1}r + h_n}{k_{n+1}r + k_n} = f_{n+2}(r) = \frac{a}{b} .$$

Letting $r = y/x$, this leads to equations

$$\begin{aligned} h_n x + h_{n+1} y &= a \\ k_n x + k_{n+1} y &= b \end{aligned} \qquad \begin{bmatrix} h_n & h_{n+1} \\ k_n & k_{n+1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} .$$

Let $M = \begin{bmatrix} h_n & h_{n+1} \\ k_n & k_{n+1} \end{bmatrix}$. We know that $\det M = \pm 1$, so M^{-1} has integer entries, and $\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} a \\ b \end{bmatrix}$, so there is a unique solution (x, y), and $x, y \in \mathbb{Z}$.

Claim. $x \neq 0$, and $x > 0 \iff y \leq 0$.

Proof.

Case I. If $y \leq 0$, then $k_n x = b - k_{n+1} y \geq b > 0$, so $x > 0$.

Case II. If $y > 0$, then $b < k_{n+1} \leq k_{n+1} y$, so $k_n x = b - k_{n+1} y < 0$; therefore $x < 0$. □

Now since ξ is between r_n and r_{n+1}, $k_n \xi - h_n$ and $k_{n+1}\xi - h_{n+1}$ have opposite signs. Therefore $x(k_n \xi - h_n)$ and $y(k_{n+1}\xi - h_{n+1})$ are either both ≥ 0 or both ≤ 0.

Adding them gives

$$x(k_n \xi - h_n) + y(k_{n+1}\xi - h_{n+1}) = (k_n x + k_{n+1} y)\xi - (h_n x + h_{n+1} y) = b\xi - a ;$$

therefore

$$\begin{aligned} |b\xi - a| &= |x(k_n \xi - h_n) + y(k_{n+1}\xi - h_{n+1})| \\ &\overset{!!}{=} |x(k_n \xi - h_n)| + |y(k_{n+1}\xi - h_{n+1})| \\ &\geq |x(k_n \xi - h_n)| \\ &\geq |k_n \xi - h_n| . \end{aligned}$$

□