Throughout today’s class:

\[ a_0, a_1, \ldots \in \mathbb{Z} \quad \text{and} \quad a_i > 0 \quad \text{for all} \quad i > 0. \]

**Lemma.** Let \( \theta = \langle a_0, a_1, \ldots \rangle \) and \( \theta_1 = \langle a_1, a_2, \ldots \rangle \). Then:

(a). \( \theta \geq a_0 \),
(b). \( \theta = a_0 + \frac{1}{\theta_1} \),
(c). \( \theta_1 > 1 \), and
(d). \( a_0 = \lfloor \theta \rfloor \).

**Theorem (Uniqueness).** Let \( \langle a_0, a_1, \ldots \rangle \) and \( \langle b_0, b_1, \ldots \rangle \) be infinite simple continued fractions. If \( \langle a_0, a_1, \ldots \rangle = \langle b_0, b_1, \ldots \rangle \), then \( a_i = b_i \) for all \( i \).

**Proof.** Induction on \( i \). Let \( \theta = \langle a_0, a_1, \ldots \rangle = \langle b_0, b_1, \ldots \rangle \). Then \( a_0 = \lfloor \theta \rfloor = b_0 \); combining this with

\[ a_0 + \frac{1}{\langle a_1, a_2, \ldots \rangle} = \theta = b_0 + \frac{1}{\langle b_1, b_2, \ldots \rangle} \]

gives \( \langle a_1, a_2, \ldots \rangle = \langle b_1, b_2, \ldots \rangle \). Repeat to get \( a_i = b_i \) for all \( i \) by induction. \( \square \)

This suggests that a procedure, similar to that for finding finite simple continued fractions, would be useful for finding infinite simple continued fractions (or at least initial finite sequences of their partial quotients \( a_i \)).

**Summarizing (so far)**

Last time we defined a function

\[ f: \{ \text{infinite integer sequences} \; a_0, a_1, \ldots \; \text{with} \; a_i > 0 \; \text{for all} \; i > 0 \} \rightarrow \mathbb{R} \setminus \mathbb{Q} \]

given by

\[ (a_0, a_1, \ldots) \mapsto \langle a_0, a_1, \ldots \rangle. \]

Now we’ve shown that this function is injective.

**Theorem (Existence).** Let \( \xi \in \mathbb{R} \) be an irrational number. Then:

(a). there exists an integer sequence \( a_0, a_1, \ldots \) with \( a_i > 0 \) for all \( i > 0 \), and irrational \( \xi_0, \xi_1, \ldots \in \mathbb{R} \), such that: (i)

\[ \langle a_0, \ldots, a_{i-1}, \xi_i \rangle = \xi \quad (\ast) \]

for all \( i \in \mathbb{N} \); and (ii) \( \xi_i > 1 \) for all \( i > 0 \).

(b). \( \langle a_0, a_1, \ldots \rangle = \xi \).
Proof. (a) By induction on \( n \in \mathbb{N} \), we will construct \( a_0, \ldots, a_{n-1} \) and \( \xi_0, \ldots, \xi_n \) such that (*) holds for all \( i \leq n \).

**Base case:** If \( n = 0 \), then no \( a_i \) need to be constructed, and we let \( \xi_0 = \xi \). Then (*) with \( i = 0 \) is \( \langle \xi_0 \rangle = \xi_0 = \xi \), so we’re done.

**Inductive step:** Assume that \( n > 0 \), and that we have \( a_0, \ldots, a_{n-2} \in \mathbb{Z} \) and irrational \( \xi_0, \ldots, \xi_{n-1} \), such that \( a_i > 0 \) for all \( 0 < i \leq n - 2 \) and (*) holds for all \( 0 \leq i \leq n - 1 \).

Let \( a_{n-1} = [\xi_{n-1}] \) and \( \xi_n = \frac{1}{\xi_{n-1} - a_{n-1}} \). Note that:

- \( \xi_n \) is irrational because \( \xi_{n-1} \) is,
- \( 0 < \xi_{n-1} - a_{n-1} < 1 \) (\( \xi_{n-1} \neq a_{n-1} \) because \( \xi_{n-1} \) is irrational),
- \( \xi_n > 1 \),
- \( \langle a_0, \ldots, a_{n-1}, \xi_n \rangle = \langle a_0, \ldots, a_{n-2}, a_{n-1} + 1/\xi_n \rangle = \langle a_0, \ldots, a_{n-2}, \xi_{n-1} \rangle = \xi \); and
- \( a_{n-1} > 0 \) if \( n > 1 \) (because \( \xi_{n-1} > 1 \) if \( n > 1 \)).

This proves (a).

(b) Let \( h_n, k_n \) (\( n \geq -2 \)) be as before, and for all \( n \in \mathbb{N} \) let

\[
r_n = \frac{h_n}{k_n} = \langle a_0, \ldots, a_n \rangle.
\]

Let \( \theta = \lim_{n \to \infty} r_n = \langle a_0, a_1, \ldots \rangle \). We need to show that \( \theta = \xi \).

For all \( n \geq 2 \) let \( f_n : \mathbb{R}_{\geq 0} \to \mathbb{R} \) be the function

\[
f_n(x) = \langle a_0, \ldots, a_{n-1}, x \rangle = \frac{h_{n-1}x + h_{n-2}}{k_{n-1}x + k_{n-2}}.
\]

Then

\[
f_n'(x) = \frac{d}{dx} \left( \frac{h_{n-1}x + h_{n-2}}{k_{n-1}x + k_{n-2}} \right)
= \frac{h_{n-1}(k_{n-1}x + k_{n-2}) - k_{n-1}(h_{n-1}x + h_{n-2})}{(k_{n-1}x + k_{n-2})^2}
= \frac{h_{n-1}k_{n-2} - k_{n-1}h_{n-2}}{(k_{n-1}x + k_{n-2})^2} = \frac{(-1)^n}{(k_{n-1}x + k_{n-2})^2}.
\]

Since \( n \geq 2 \), \( k_{n-1} > 0 \) and \( k_{n-2} > 0 \), so \( k_{n-1}x + k_{n-2} > 0 \) for all \( x > 0 \); therefore \( f_n \) is differentiable on \((0, \infty)\). In fact, it is monotone there: increasing if \( n \) is even, or decreasing if \( n \) is odd. (This generalizes Exercise 7.1.5 if \( n \geq 2 \).

Therefore \( f_n(\xi_n) = \langle a_0, \ldots, a_{n-1}, \xi_n \rangle = \xi \) is between

\[
\lim_{x \to 0^+} f_n(x) = \frac{h_{n-2}}{k_{n-2}} = r_{n-2} \quad \text{and} \quad \lim_{x \to \infty} f_n(x) = \frac{h_{n-1}}{k_{n-1}} = r_{n-1}
\]
(i.e., \( r_{n-2} < \xi < r_{n-1} \) or \( r_{n-1} < \xi < r_{n-2} \)).
In particular, \(|\xi - r_{n-1}| < |r_{n-1} - r_{n-2}|\). Since \(|r_{n-1} - r_{n-2}| \to 0\) as \(n \to \infty\), we have \(\lim_{n \to \infty} |\xi - r_n| = 0\), so \(\lim_{n \to \infty} r_n = \xi\); therefore \(\xi = \theta\). \(\Box\)

**Corollary.** \(\xi\) is in the image of the map \(f\) defined earlier, so \(f\) is surjective. Therefore \(f\) is bijective.

Summarizing, we have bijections:

\[
\mathbb{Q} = \{\text{rational numbers}\} \leftrightarrow \{\text{finite simple continued fractions} \langle a_0, \ldots, a_n \rangle \text{ with } n = 0 \text{ or } a_n > 1 \} \\
\leftrightarrow \{\text{finite simple continued fractions} \langle a_0, \ldots, a_n \rangle \text{ with } n > 0 \text{ and } a_n = 1\}
\]

and

\[
\mathbb{R} \setminus \mathbb{Q} = \{\text{irrational real numbers}\} \\
\leftrightarrow \{\text{infinite simple continued fractions}\}.
\]

**Example Computations**

(1) Let \(\xi = \sqrt{10}\). Then

\[
a_0 = \lceil \sqrt{10} \rceil = 3; \quad \xi_1 = \frac{1}{\sqrt{10} - 3} = \frac{\sqrt{10} + 3}{10 - 3^2} = \sqrt{10} + 3
\]

\[
a_1 = \lceil \sqrt{10} + 3 \rceil = 6; \quad \xi_2 = \frac{1}{(\sqrt{10} + 3) - 6} = \frac{1}{\sqrt{10} - 3} = \sqrt{10} + 3 = \xi_1
\]

therefore \(\sqrt{10} = \langle 3, 6, 6, \ldots \rangle\).

(2) Let \(\xi = \sqrt{6}\). Then

\[
a_0 = \lceil \sqrt{6} \rceil = 2; \quad \xi_1 = \frac{1}{\sqrt{6} - 2} = \frac{\sqrt{6} + 2}{6 - 2^2} = \frac{\sqrt{6} + 2}{2}
\]

\[
a_1 = \lceil \frac{\sqrt{6} + 2}{2} \rceil = 2; \quad \xi_2 = \frac{1}{\frac{\sqrt{6} + 2}{2} - 2} = \frac{2}{\sqrt{6} + 2 - 4} = \frac{2}{\sqrt{6} - 2}
\]

\[
= \frac{2(\sqrt{6} + 2)}{6 - 4} = \sqrt{6} + 2
\]

\[
a_2 = \lceil \sqrt{6} + 2 \rceil = 4; \quad \xi_3 = \frac{1}{\sqrt{6} + 2 - 4} = \frac{1}{\sqrt{6} - 2} = \xi_1.
\]
(3) In the opposite direction, what is \(1, 2, 1, 3, 1, 3, \ldots\) ?
First find \(1, 3, 1, 3, \ldots\). Let \(\theta = 1, 3, 1, 3, \ldots\). Then

\[
\theta = 1 + \frac{1}{3 + \frac{1}{\theta}} = 1 + \frac{\theta}{3\theta + 1} = \frac{4\theta + 1}{3\theta + 1}.
\]

Therefore \(\theta(3\theta + 1) - (4\theta + 1) = 0\); \(3\theta^2 - 3\theta - 1 = 0\); so

\[
\theta = \frac{3 \pm \sqrt{9 + 12}}{6} = \frac{3 \pm \sqrt{21}}{6}.
\]

Since \(\theta > 1\), it can’t be \(< \frac{1}{2}\), so it must be \(\frac{3 + \sqrt{21}}{6}\).

Then

\[
\langle 1, 2, 1, 3, 1, 3, \ldots \rangle = 1 + \frac{1}{2 + \frac{1}{\frac{3 + \sqrt{21}}{6}}},
\]

whatever that is.

(4) Euler showed that

\[
e = \langle 2, 1, 2, 1, 4, 1, 1, 6, 1, 1, \ldots \rangle.
\]

**Why Do We Study Continued Fractions?**

Answer: *Diophantine approximation.*

A large partial quotient \(a_n\) indicates that (the rational number) \(\langle a_0, \ldots, a_{n-1} \rangle\) is very close to the number \(\langle a_0, a_1, \ldots \rangle\) or \(\langle a_0, \ldots, a_k \rangle\) (\(k \geq n\)).

Example.

\[
\frac{4}{7} = \langle 0, 1, 1, 3 \rangle
\]

\[
\frac{4}{7} \approx 0.571 = \langle 0, 1, 1, 3, 47, 3 \rangle
\]

\[
\frac{4}{7} \approx 0.572 = \langle 0, 1, 1, 2, 1, 35 \rangle
\]