Definition. Let \( n \in \mathbb{N} \) and \( x_0, \ldots, x_n \in \mathbb{R} \) with \( x_i > 0 \) for all \( i > 0 \). Then

\[
\langle x_0, \ldots, x_n \rangle = \begin{cases} 
  x_0 & \text{if } n = 0; \\
  x_0 + \frac{1}{\langle x_1, \ldots, x_n \rangle} & \text{if } n > 0.
\end{cases}
\]

Some basic properties:

1. \( \langle x_0, x_1, \ldots, x_n \rangle = x_0 + \frac{1}{\langle x_1, \ldots, x_n \rangle} \) if \( n > 0 \) (by definition).
2. \( \langle x_0, \ldots, x_n \rangle = \langle x_0, \ldots, x_{n-2}, x_{n-1} + \frac{1}{x_n} \rangle \) if \( n > 0 \) (by induction)
3. \( \langle x_0, \ldots, x_n \rangle \geq x_0 \), with equality \( \iff n = 0 \).

Proof of (3). This is proved by induction on \( n \geq 0 \).

Base case: If \( n = 0 \) then it’s easy.

Inductive step: If \( n > 0 \) and it’s true for \( n - 1 \), then

\[
\langle x_1, \ldots, x_n \rangle \geq x_1 > 0,
\]

so \( 1/\langle x_1, \ldots, x_n \rangle > 0 \). Therefore

\[
\langle x_0, x_1, \ldots, x_n \rangle = x_0 + \frac{1}{\langle x_1, \ldots, x_n \rangle} > x_0.
\]

Definition. A finite continued fraction is simple if \( x_0, \ldots, x_n \in \mathbb{Z} \).

Every finite simple continued fraction evaluates to a rational number.

We’ll show: Every rational number can be written as a finite simple continued fraction in exactly two ways.

(Later we’ll see: Every (real) irrational number can be written uniquely as an infinite simple continued fraction.)

Theorem (Existence). For all \( x \in \mathbb{Q} \) there is a finite sequence \( a_0, \ldots, a_n \in \mathbb{Z} \) with \( n \in \mathbb{N} \) such that \( a_i > 0 \) for all \( i > 0 \) and \( x = \langle a_0, \ldots, a_n \rangle \).

Proof. ...