Math 115. Slides from the Lecture of September 29

This handout contains the slides from the lecture of September 29 (with a more complete Division Algorithm).

Polynomials with Coefficients in $\mathbb{C}$

The main line of today's class will mimic the following statements and proofs for polynomials with coefficients in $\mathbb{C}$ (or $\mathbb{R}$ or $\mathbb{Q}$).

Definition. $\mathbb{C}[x]$ is the set of polynomials with coefficients in $\mathbb{C}$. $\mathbb{R}[x]$ and $\mathbb{Q}[x]$ are defined analogously.

We prove here that a nonzero polynomial in $\mathbb{C}[x]$ of degree n has at most n roots (in $\mathbb{C}$). (In fact, it has exactly n roots, when counted with multiplicities, but this will not be proved for congruences modulo p.)

For the rest of today's class, we will use the convention that the zero polynomial in $\mathbb{C}[x]$ has degree $-\infty$.

Theorem (Division Algorithm for Polynomials in $\mathbb{C}[x]$). *Let $f, g \in \mathbb{C}[x]$ with $g \neq 0$. Then there are polynomials $q, r \in \mathbb{C}[x]$ such that $f(x) = q(x)g(x) + r(x)$ and $\deg r < \deg g$. Also, q and r are unique with these properties.*

Proof. **Existence.** Long division of polynomials.

Uniqueness. Suppose that $q_2, r_2 \in \mathbb{C}[x]$ also satisfy $f(x) = q_2(x)g(x) + r_2(x)$ and $\deg r_2 < \deg g$. Then

$$r(x) - r_2(x) = (q_2(x) - q(x))g(x) \ . \qquad (*)$$

But now the left-hand side has degree $< \deg g$, and the only way this can happen is for the factor $q_2 - q$ to be zero. Therefore $q_2 = q$. Therefore the left-hand side of (*) must be zero, so $r_2 = r$ also. This gives uniqueness. □

Corollary. *Let $f \in \mathbb{C}[x]$ and $a \in \mathbb{C}$. Then a is a root of f (i.e., $f(a) = 0$) if and only if $(x - a) \mid f$ (i.e., $f(x) = (x - a)g(x)$ for some $g \in \mathbb{C}[x]$).*

Proof. Write $f(x) = (x - a)g(x) + r(x)$ with $\deg r < 1$. Then r is a constant c, and substituting $x = a$ gives $f(a) = (a - a)g(a) + c = c$ (because $a - a = 0$), so $f(a) = c$. Therefore

$$f(a) = 0 \iff c = 0 \iff (x - a) \mid f \ .$$

□

Corollary. *If $f \in \mathbb{C}[x]$ and $a_1, \ldots, a_r \in \mathbb{C}$ are distinct roots of f (with $r > 0$), then writing $f(x) = (x - a_1)g(x)$, we have that $a_2, \ldots, a_r$ are distinct roots of g.*

Proof. Exercise. □