Math 115. Slides from the Lecture of September 29

This handout contains the slides from the lecture of September 29 (with a more complete Division Algorithm).

Polynomials with Coefficients in \( \mathbb{C} \)

The main line of today’s class will mimic the following statements and proofs for polynomials with coefficients in \( \mathbb{C} \) (or \( \mathbb{R} \) or \( \mathbb{Q} \)).

**Definition.** \( \mathbb{C}[x] \) is the set of polynomials with coefficients in \( \mathbb{C} \). \( \mathbb{R}[x] \) and \( \mathbb{Q}[x] \) are defined analogously.

We prove here that a nonzero polynomial in \( \mathbb{C}[x] \) of degree \( n \) has at most \( n \) roots (in \( \mathbb{C} \)). (In fact, it has exactly \( n \) roots, when counted with multiplicities, but this will not be proved for congruences modulo \( p \).)

For the rest of today’s class, we will use the convention that the zero polynomial in \( \mathbb{C}[x] \) has degree \( -\infty \).

**Theorem** (Division Algorithm for Polynomials in \( \mathbb{C}[x] \)). Let \( f, g \in \mathbb{C}[x] \) with \( g \neq 0 \). Then there are polynomials \( q, r \in \mathbb{C}[x] \) such that \( f(x) = q(x)g(x) + r(x) \) and \( \deg r < \deg g \). Also, \( q \) and \( r \) are unique with these properties.

**Proof.**

**Existence.** Long division of polynomials.

**Uniqueness.** Suppose that \( q_2, r_2 \in \mathbb{C}[x] \) also satisfy \( f(x) = q_2(x)g(x) + r_2(x) \) and \( \deg r_2 < \deg g \). Then

\[
  r(x) - r_2(x) = (q_2(x) - q(x))g(x) .
\]

But now the left-hand side has degree < \( \deg g \), and the only way this can happen is for the factor \( q_2 - q \) to be zero. Therefore \( q_2 = q \). Therefore the left-hand side of (*) must be zero, so \( r_2 = r \) also. This gives uniqueness. \( \square \)

**Corollary.** Let \( f \in \mathbb{C}[x] \) and \( a \in \mathbb{C} \). Then \( a \) is a root of \( f \) (i.e., \( f(a) = 0 \)) if and only if \( (x - a) | f \) (i.e., \( f(x) = (x - a)g(x) \) for some \( g \in \mathbb{C}[x] \)).

**Proof.** Write \( f(x) = (x - a)g(x) + r(x) \) with \( \deg r < 1 \). Then \( r \) is a constant \( c \), and substituting \( x = a \) gives \( f(a) = (a - a)g(a) + c = c \) (because \( a - a = 0 \)), so \( f(a) = c \). Therefore

\[
  f(a) = 0 \iff c = 0 \iff (x - a) | f .
\]

**Corollary.** If \( f \in \mathbb{C}[x] \) and \( a_1, \ldots, a_r \in \mathbb{C} \) are distinct roots of \( f \) (with \( r > 0 \)), then writing \( f(x) = (x - a_1)g(x) \), we have that \( a_2, \ldots, a_r \) are distinct roots of \( g \).

**Proof.** Exercise. \( \square \)