Math 115. Three Types of Induction

This handout discusses three types of induction, given below, and shows that they are equivalent. These types are:

Principle of Induction. *Let* $P(n)$ *be a statement (proposition) depending on a number* $n \in \mathbb{N}$ *. Assume that*

(i). *(basis statement)* $P(0)$ *is true, and*
(ii). *(inductive step) for all* $n \in \mathbb{N}$ *, if* $P(n)$ *is true, then* $P(n+1)$ *is also true.*

Then $P(n)$ *is true for all* $n \in \mathbb{N}$ *.*

Principle of Strong Induction. *Let* $P(n)$ *be a statement (proposition) depending on a number* $n \in \mathbb{N}$ *. Assume that*

(i). *(basis statement)* $P(0)$ *is true, and*
(ii′). *(inductive step) for all* $n \in \mathbb{N}$ *, if* $P(k)$ *is true for all* $k = 0, 1, \dots, n$ *, then* $P(n+1)$ *is also true.*

Then $P(n)$ *is true for all* $n \in \mathbb{N}$ *.*

Well-Ordering of $\mathbb{N}$ **.** *Every nonempty subset of* $\mathbb{N}$ *has a smallest element.*

A. Strong Induction implies Induction

Since $n \ge 0$, we have

$$P(k) \text{ is true for all } k = 0, 1, \dots, n \qquad \text{implies} \qquad P(n) \text{ is true .}$$

Therefore condition (ii) implies condition (ii′). This is because if (ii) is true and if $P(k)$ is true for all $k = 0, 1, \dots, n$, then $P(n)$ is true, and therefore by (ii) $P(n+1)$ is true. This verifies (ii′).

By a similar argument, the Principal of Strong Induction implies the Principle of Induction, because hypothesis (ii′) is weaker.

B. Induction implies Strong Induction

Let $Q(n)$ be the statement " $P(k)$ is true for all $k = 0, 1, \dots, n$." Then condition (i) for P implies condition (i) for Q . Also, assume that condition (ii′) holds for P . Then condition (ii) holds for $Q(n)$, because the truth of $Q(n)$ implies that $P(n+1)$ is true (by (ii′)), and then $Q(n+1)$ is true because both $Q(n)$ and $P(n+1)$ are true.

By induction, it then follows that $Q(n)$ is true for all n , so $P(n)$ is also true for all n .

C. Well-Ordering of $\mathbb{N}$ implies Induction

We will prove this by contradiction. (This is called the "method of the minimal counterexample.")

Let P be a proposition as in the statement of the Principle of Induction. Let S be the set of all $n \in \mathbb{N}$ such that $P(n)$ is false, and assume by way of contradiction that S is nonempty.

Then S has a smallest element by the well ordering property; call it m. We know by assumption (i) that $m \neq 0$. Then $m-1 \in \mathbb{N}$, and since m is the smallest element of S, we have $m-1 \notin S$. Therefore $P(m-1)$ is true; therefore $P(m)$ is also true by assumption (ii). This implies that $m \notin S$, which gives a contradiction.

Thus, if $P(n)$ is a proposition depending on $n \in \mathbb{N}$, and if P satisfies assumptions (i) and (ii) of the Principle of Induction, then $P(n)$ is true for all $n \in \mathbb{N}$.

D. Strong Induction implies Well-Ordering of $\mathbb{N}$

Suppose that the Principle of Strong Induction is true.

In order to prove that $\mathbb{N}$ is well ordered, we prove the following statement $P(n)$ by strong induction on n: If S is a subset of $\mathbb{N}$ and $n \in S$, then S has a smallest element.

The basis step is true, because if $0 \in S$ then 0 is the smallest element of S, because there are no smaller elements of $\mathbb{N}$.

For the inductive step, suppose that $P(k)$ is true for all $k = 0, 1, \ldots, n$. To show that $P(n+1)$ is true, let S be a subset of $\mathbb{N}$ that contains $n+1$. If $n+1$ is the smallest element of S, then we're done. Otherwise, S has a smaller element k, and $P(k)$ is true by the inductive assumption, so again S has a smallest element.

Therefore, by strong induction, $P(n)$ is true for all $n \in \mathbb{N}$. This implies the well-ordering of $\mathbb{N}$, because if S is a nonempty subset of $\mathbb{N}$, then pick $n \in S$. Since $n \in \mathbb{N}$, $P(n)$ is true, and therefore S has a smallest element.

Comments

1. Parts C and D make part A redundant.

2. Part D is not really relevant, since well-ordering of $\mathbb{N}$ is an axiom, but induction is a theorem. However, it may be useful if one has an axiom system in which induction is an axiom instead of well-ordering on $\mathbb{N}$. It is also useful for understanding why induction may or may not be valid for sets other than $\mathbb{N}$.

3. We also have principles of induction with $\mathbb{N}$ replaced by $\mathbb{Z}_{>0}$. Moreover, for any $n_0 \in \mathbb{Z}$, we can do induction on the set of all integers $n \geq n_0$, merely by letting $P(k)$ be the proposition that whatever is true for $k + n_0$.

4. In the statement of Strong Induction, (i) and (ii$'$) can be combined into the following statement:

(ii$''$). (inductive step) for all $n \in \mathbb{N}$, if $P(k)$ is true for all $k \in \mathbb{N}$ with $k < n$, then $P(n)$ is also true.

Indeed, when $n = 0$ the hypothesis is vacuously true, which gives (i), and when $n > 1$ the condition is equivalent to (ii$'$) for $n - 1$.