Math 115. Slides from the Lecture of November 26

(More) Infinite Simple Continued Fractions

Throughout today's class:

$$a_0, a_1, \dots \in \mathbb{Z} \quad \text{and} \quad a_i > 0 \text{ for all } i > 0 \ .$$

$$\begin{aligned} &h_{-2} = 0\ , \quad h_{-1} = 1\ , \quad h_n = a_n h_{n-1} + h_{n-2} \quad \text{for all } n \ge 0 \\ &k_{-2} = 1\ , \quad k_{-1} = 0\ , \quad k_n = a_n k_{n-1} + k_{n-2} \end{aligned}$$

$$r_n = \langle a_0, \dots, a_n \rangle = \frac{h_n}{k_n} \qquad \text{for all } n \ge 0 \ .$$

[stuff on blackboard]

Lemma. *Let* $\theta = \langle a_0, a_1, \dots \rangle$ *and* $\theta_1 = \langle a_1, a_2, \dots \rangle$ *. Then:*

(a). $\theta > a_0$,
(b). $\theta = a_0 + \dfrac{1}{\theta_1}$,
(c). $\theta_1 > 1$, *and*
(d). $a_0 = [\theta]$.

Proof. (a). $\theta > r_0 = \langle a_0 \rangle = a_0$.

(b). By continuity of $a_0 + 1/x$ on $(0, \infty)$, we have

$$\begin{aligned} \theta &= \lim_{n\to\infty} \langle a_0, \dots, a_n \rangle = \lim_{n\to\infty} \left(a_0 + \frac{1}{\langle a_1, \dots, a_n \rangle} \right) \\ &= a_0 + \frac{1}{\lim_{n\to\infty} \langle a_1, \dots, a_n \rangle} = a_0 + \frac{1}{\theta_1} \ . \end{aligned}$$

(c). By (b) and the fact that $\langle a_2, a_3, \dots \rangle \ge a_2 > 0$,

$$\theta_1 = a_1 + \frac{1}{\langle a_2, a_3, \dots \rangle} > a_1 \ge 1 \ .$$

(d). By (a), (b), and (c) (respectively),

$$a_0 \le \theta = a_0 + \frac{1}{\theta_1} < a_0 + 1 \ .$$

□

Theorem (Uniqueness). *Let* $\langle a_0, a_1, \dots \rangle$ *and* $\langle b_0, b_1, \dots \rangle$ *be infinite simple continued fractions. If* $\langle a_0, a_1, \dots \rangle = \langle b_0, b_1, \dots \rangle$ *, then* $a_i = b_i$ *for all* i *.*

Proof. Induction on i. Let $\theta = \langle a_0, a_1, \dots \rangle = \langle b_0, b_1, \dots \rangle$. Then $a_0 = [\theta] = b_0$; combining this with

$$a_0 + \frac{1}{\langle a_1, a_2, \dots \rangle} = \theta = b_0 + \frac{1}{\langle b_1, b_2, \dots \rangle}$$

gives $\langle a_1, a_2, \dots \rangle = \langle b_1, b_2, \dots \rangle$. Repeat to get $a_i = b_i$ for all i by induction. □

This suggests that a procedure, similar to that for finding finite simple continued fractions, would be useful for finding infinite simple continued fractions (or at least initial finite sequences of their partial quotients a_i).

Summarizing (so far)

We defined a function

$$f\colon \{\text{infinite integer sequences } a_0, a_1, \dots \text{ with } a_i > 0 \text{ for all } i > 0\} \longrightarrow \mathbb{R} \setminus \mathbb{Q}$$

given by

$$(a_0, a_1, \dots) \mapsto \langle a_0, a_1, \dots \rangle .$$

We showed that this function is *injective*.

Theorem (Existence). *Let $\xi \in \mathbb{R}$ be an irrational number. Then:*

(a). *there exists an integer sequence $a_0, a_1, \dots$ with $a_i > 0$ for all $i > 0$, and irrational $\xi_0, \xi_1, \dots \in \mathbb{R}$, such that: (i)*

$$\langle a_0, \dots, a_{i-1}, \xi_i \rangle = \xi \qquad (*)$$

for all $i \in \mathbb{N}$; and (ii) $\xi_i > 1$ for all $i > 0$.

(b). $\langle a_0, a_1, \dots \rangle = \xi$.

Proof. (a). By induction on $n \in \mathbb{N}$, we will construct $a_0, \dots, a_{n-1}$ and $\xi_0, \dots, \xi_n$ such that (*) holds for all $i \le n$.

Base case: If $n = 0$, then no a_i need to be constructed, and we let $\xi_0 = \xi$. Then (*) with $i = 0$ is $\langle \xi_0 \rangle = \xi_0 = \xi$, so we're done.

Inductive step: Assume that $n > 0$, and that we have $a_0, \dots, a_{n-2} \in \mathbb{Z}$ and irrational $\xi_0, \dots, \xi_{n-1}$, such that $a_i > 0$ for all $0 < i \le n-2$ and (*) holds for all $0 \le i < n$.

Let $a_{n-1} = [\xi_{n-1}]$ and $\xi_n = \dfrac{1}{\xi_{n-1} - a_{n-1}}$. Note that:

- ξ_n is irrational because ξ_{n-1} is,
- $0 < \xi_{n-1} - a_{n-1} < 1$ ($\xi_{n-1} \ne a_{n-1}$ because ξ_{n-1} is irrational),
- $\xi_n > 1$,
- $\langle a_0, \dots, a_{n-1}, \xi_n \rangle = \langle a_0, \dots, a_{n-2}, a_{n-1} + 1/\xi_n \rangle = \langle a_0, \dots, a_{n-2}, \xi_{n-1} \rangle = \xi$; and
- $a_{n-1} > 0$ if $n > 1$ (because $\xi_{n-1} > 1$ if $n > 1$).

This proves (a).

(b). Let $\theta = \lim_{n\to\infty} r_n = \langle a_0, a_1, \dots \rangle$. We need to show that $\theta = \xi$.

For all $n \ge 2$ let $f_n \colon \mathbb{R}_{\ge 0} \to \mathbb{R}$ be the function

$$f_n(x) = \langle a_0, \dots, a_{n-1}, x \rangle = \frac{h_{n-1}x + h_{n-2}}{k_{n-1}x + k_{n-2}} .$$

Then

$$\begin{aligned} f_n'(x) &= \frac{d}{dx}\left(\frac{h_{n-1}x + h_{n-2}}{k_{n-1}x + k_{n-2}}\right) \\ &= \frac{h_{n-1}(k_{n-1}x + k_{n-2}) - k_{n-1}(h_{n-1}x + h_{n-2})}{(k_{n-1}x + k_{n-2})^2} \\ &= \frac{h_{n-1}k_{n-2} - k_{n-1}h_{n-2}}{(k_{n-1}x + k_{n-2})^2} = \frac{(-1)^n}{(k_{n-1}x + k_{n-2})^2} . \end{aligned}$$

Since $n \ge 2$, $k_{n-1} > 0$ and $k_{n-2} > 0$, so $k_{n-1}x + k_{n-2} > 0$ for all $x > 0$; therefore f_n is differentiable on $(0, \infty)$. In fact, it is *monotone* there: increasing if n is even, or decreasing if n is odd. (This generalizes Exercise 7.1.5 if $n \ge 2$.)

Therefore $f_n(\xi_n) = \langle a_0, \dots, a_{n-1}, \xi_n \rangle = \xi$ is *between*

$$\lim_{x\to 0^+} f_n(x) = \frac{h_{n-2}}{k_{n-2}} = r_{n-2} \qquad \text{and} \qquad \lim_{x\to\infty} f_n(x) = \frac{h_{n-1}}{k_{n-1}} = r_{n-1}$$

(i.e., $r_{n-2} < \xi < r_{n-1}$ or $r_{n-1} < \xi < r_{n-2}$).

In particular, $|\xi - r_{n-1}| < |r_{n-1} - r_{n-2}|$. Since $|r_{n-1} - r_{n-2}| \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} |\xi - r_n| = 0$, so $\lim_{n\to\infty} r_n = \xi$; therefore $\xi = \theta$. □

Corollary. *ξ is in the image of the map f defined earlier, so f is surjective. Therefore f is bijective.*

Summarizing, we have bijections:

$$\begin{aligned} \mathbb{Q} = \{\text{rational numbers}\} &\leftrightarrow \{\text{finite simple continued fractions} \\ &\qquad \langle a_0, \dots, a_n \rangle \text{ with } n = 0 \text{ or } a_n > 1\} \\ &\leftrightarrow \{\text{finite simple continued fractions} \\ &\qquad \langle a_0, \dots, a_n \rangle \text{ with } n > 0 \text{ and } a_n = 1\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{R} \setminus \mathbb{Q} &= \{\text{irrational real numbers}\} \\ &\leftrightarrow \{\text{infinite simple continued fractions}\} . \end{aligned}$$

Example Computations

(1) Let $\xi = \sqrt{10}$. Then

$$a_0 = [\sqrt{10}] = 3\,; \quad \xi_1 = \frac{1}{\sqrt{10}-3} = \frac{\sqrt{10}+3}{10-3^2} = \sqrt{10}+3$$
$$a_1 = [\sqrt{10}+3] = 6\,; \quad \xi_2 = \frac{1}{(\sqrt{10}+3)-6} = \frac{1}{\sqrt{10}-3} = \sqrt{10}+3 = \xi_1\,;$$

therefore $\sqrt{10} = \langle 3, 6, 6, \dots \rangle$.

(2) Let $\xi = \sqrt{6}$. Then

$$a_0 = [\sqrt{6}] = 2\,; \quad \xi_1 = \frac{1}{\sqrt{6}-2} = \frac{\sqrt{6}+2}{6-2^2} = \frac{\sqrt{6}+2}{2}$$
$$a_1 = \left[\frac{\sqrt{6}+2}{2}\right] = 2\,; \quad \xi_2 = \frac{1}{\frac{\sqrt{6}+2}{2}-2} = \frac{2}{\sqrt{6}+2-4} = \frac{2}{\sqrt{6}-2}$$
$$= \frac{2(\sqrt{6}+2)}{6-4} = \sqrt{6}+2$$
$$a_2 = [\sqrt{6}+2] = 4\,; \quad \xi_3 = \frac{1}{\sqrt{6}+2-4} = \frac{1}{\sqrt{6}-2} = \xi_1\,;$$

therefore $\sqrt{6} = \langle 2, 2, 4, 2, 4, \dots \rangle$.

(3) In the opposite direction, what is $\langle 1, 2, 1, 3, 1, 3, \dots \rangle$?

First find $\langle 1, 3, 1, 3, \dots \rangle$. Let $\theta = \langle 1, 3, 1, 3, \dots \rangle$. Then

$$\theta = 1 + \frac{1}{3+\frac{1}{\theta}} = 1 + \frac{\theta}{3\theta+1} = \frac{4\theta+1}{3\theta+1}\,.$$

Therefore $\theta(3\theta+1) - (4\theta+1) = 0$; $3\theta^2 - 3\theta - 1 = 0$; so

$$\theta = \frac{3 \pm \sqrt{9+12}}{6} = \frac{3 \pm \sqrt{21}}{6}\,.$$

Since $\theta > 1$, it can't be $< \frac{1}{2}$, so it must be $\frac{3+\sqrt{21}}{6}$.

Then

$$\langle 1, 2, 1, 3, 1, 3, \dots \rangle = 1 + \frac{1}{2+\frac{1}{\frac{3+\sqrt{21}}{6}}}\,,$$

whatever that is.

(4) Euler showed that

$$e = \langle 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots \rangle \ .$$

Why Do We Study Continued Fractions?

Answer: *Diophantine approximation.*

A large **partial quotient** a_n indicates that (the rational number) $\langle a_0, \dots, a_{n-1} \rangle$ is very close to the number $\langle a_0, a_1, \dots \rangle$ or $\langle a_0, \dots, a_k \rangle$ ($k \geq n$).

Example.

$$\frac{4}{7} = \langle 0, 1, 1, 3 \rangle$$
$$\frac{4}{7} \approx 0.571 = \langle 0, 1, 1, 3, 47, 3 \rangle$$
$$\frac{4}{7} \approx 0.572 = \langle 0, 1, 1, 2, 1, 35 \rangle$$